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Uniform Yomdin–Gromov parametrizations and points of bounded height in valued fields

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We prove a uniform version of non-Archimedean Yomdin–Gromov parametrizations in a definable context with algebraic Skolem functions in the residue field. The parametrization result allows us to bound the number of $\mathbb{F}_q[t]$-points of bounded degrees of algebraic varieties, uniformly in the cardinality q of the finite field $\mathbb{F}_q$ and the degree, generalizing work by Sedunova for fixed q. We also deduce a uniform non-Archimedean Pila–Wilkie theorem, generalizing work by Cluckers–Comte–Loeser.

1. Introduction

Since the pioneering work [Bombieri and Pila 1989], the determinant method of Bombieri and Pila has been used in various contexts to count integer and rational points of bounded height in algebraic or analytic varieties. Parametrization results, as initiated by Yomdin and Gromov, play a prominent role in some of the most fruitful applications of this method, such as the Pila and Wilkie counting theorem [2006] for definable sets in o-minimal structures. In the non-Archimedean setting, Cluckers, Comte and Loeser prove in [Cluckers et al. 2015] an analog of the Pila–Wilkie counting theorem, but for subanalytic sets in $\mathbb{Q}_p$, the field of p-adic numbers. Their proof relies also on a Yomdin–Gromov type parametrization result. The aim of this paper is to extend their result to obtain bounds uniform in p for some counting points of bounded height problems, over $\mathbb{Q}_p$ and over $\mathbb{F}_p((t))$. Before discussing our parametrization result, we start by presenting the applications to point counting.

1.1. ***Point counting in function fields.*** For q a prime power, consider the finite field with q elements $\mathbb{F}_q$ and for each positive integer n, let $\mathbb{F}_q[t]_n$ be the set of polynomials with coefficients in $\mathbb{F}_q$ and degree (strictly) less than n. Cilleruelo and Shparlinski [2013] have raised the question of bounding the number of $\mathbb{F}_q[t]_n$-points in plane curves. That question was settled by Sedunova [2017]. A particular case of our main theorem is a uniform version of her results. We refer to Theorem 4.1.1 for a more general statement, namely for X of arbitrary dimension. For an affine variety X defined over a subring of $\mathbb{F}_q((t))$, write $X(\mathbb{F}_q[t])_n$ for the subset of $X(\mathbb{F}_q((t)))$ consisting of points whose coordinates lie in $\mathbb{F}_q[t]_n$.

Theorem A. *Fix an integer $\delta > 0$. Then there exist real numbers $C = C(\delta)$ and $N = N(\delta)$ such that for each prime $p > N$, each $q = p^\alpha$, each integer $n > 0$ and each irreducible plane curve $X \subseteq \mathbb{A}^2_{\mathbb{F}_q((t))}$ of*

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degree δ, one has

$$\#X(\mathbb{F}_q[t])_n \le Cn^2 q^{\lceil n/\delta \rceil}.$$

A similar statement is proved by Sedunova [2017], for fixed q. More precisely, she proves that fixing δ, q and $\varepsilon > 0$, there exists a constant $C' = C'(\delta, q, \varepsilon)$ such that for each irreducible plane curve $X \subseteq \mathbb{A}^2_{\mathbb{F}_q[t]}$ of degree δ and positive integer n,

$$\#X(\mathbb{F}_q[t])_n \le C' q^{n((1/\delta)+\varepsilon)}.$$

Observe that our result improves Sedunova's by replacing the ε factor by a polylogarithmic term. By the very nature of our methods, which are model-theoretic, we are however unable to establish such a result for q a power of a small prime p.

Recently, F. Vermeulen [2020] improved our Theorem A and Sedunova's results. More precisely, he obtains a variant of Theorem A for all primes p and with, moreover, a polynomial dependence of the constant C on the degree δ.

1.2. ***A uniform non-Archimedean point counting theorem.*** We state a uniform version of the Cluckers–Comte–Loeser non-Archimedean point counting theorem. A semialgebraic set is a set defined by a first-order formula in the language $\mathcal{L}_{\mathrm{div}} = \{0, 1, +, \cdot, |\}$ and parameters in $\mathbb{Z}[[t]]$, where $|$ is a relation interpreted by $x \mid y$ if and only if $\mathrm{ord}(y) \le \mathrm{ord}(x)$, with ord the valuation. As usual, we identify definable sets with the formulas that define them. Subanalytic sets are definable sets in the language obtained by adding a new symbol for each analytic function with coefficients in $\mathbb{Z}[[t]]$ to the language $\mathcal{L}_{\mathrm{div}}$. For each local field L of characteristic zero, we fix a choice of uniformizer ϖ_L and view it as a $\mathbb{Z}[[t]]$-ring by sending t to φ_L. Hence, we can consider the L-points of a semialgebraic or subanalytic set, for L a local field of any characteristic. The notion of semialgebraic and subanalytic sets considered in Section 5 is slightly more general than the one considered here; see also Setting 3.1.1.

The dimension of a subanalytic set X is the largest d such that there exists a coordinate projection p to a linear space of dimension d such that $p(X)$ contains an open ball. A subanalytic set is said to be of pure dimension d if for each $x \in X$ and every ball B centered at x, $X \cap B$ is of dimension d. If $X \subseteq L^n$, we denote by X^{alg} the union of all semialgebraic curves of pure dimension 1 contained in X. Observe that in general, X^{alg} is not semialgebraic (nor subanalytic).

If $X \subseteq K^m$ and $H \ge 1$, with K a field of characteristic zero, we denote by $X(\mathbb{Q}, H)$ the set of $x = (x_1, \dots, x_m) \in X \cap \mathbb{Q}^m$ that can be written as $x_i = a_i/b_i$, with $a_i, b_i \in \mathbb{Z}$, $|a_i|, |b_i| \le H$ (where $|\cdot|$ is the Archimedean absolute value). If $X \subseteq L^m$, where $L = \mathbb{F}_q((t))$, we denote by $X(\mathbb{F}_q(t), H)$ the set of $x = (x_1, \dots, x_m) \in X \cap \mathbb{F}_q(t)^m$ that can be written as $x_i = a_i/b_i$, with $a_i, b_i \in \mathbb{F}_q[t]$ of degree less than or equal to $\log_q(H)$.

The following result is a particular case of Theorem 5.2.2. It provides a uniform version of Theorem 4.2.4 of [Cluckers et al. 2015].

Theorem B. *Let X be a subanalytic set of dimension m in n variables, with $m < n$. Fix $\varepsilon > 0$. Then there exists $C = C(X, \varepsilon)$, $N = N(X, \varepsilon)$, $\alpha = \alpha(n, m)$ and a semialgebraic set $W^\varepsilon \subseteq X$ such that for each*

$H \geq 1$ and each local field L, with residue field of characteristic $p_L > N$ and cardinal q_L, the following holds. We have $W^{\varepsilon}(L) \subseteq X(L)^{\mathrm{alg}}$ and if L is of characteristic zero,

$$\#(X \backslash W^{\varepsilon})(L)(\mathbb{Q}, H) \leq C(X, \varepsilon) q_L^{\alpha} H^{\varepsilon}.$$

If L is of positive characteristic, then

$$\#(X \backslash W^{\varepsilon})(L)(\mathbb{F}_{q_L}(t), H) \leq C(X, \varepsilon) q_L^{\alpha} H^{\varepsilon}.$$

An important step toward the proof of Theorem B is Proposition 5.1.4, which states that integer points of height at most H and lying in a subanalytic set X of dimension m in n variables are contained in $Cq^m \log(H)^{\alpha}$ algebraic hypersurfaces of degree $C' \log(H)^{\beta}$, where α and β are explicit constants depending only on n and m.

1.3. *Uniform Yomdin–Gromov parametrizations.* The proofs of Theorems A and B rely on the following parametrization result.

Fix a positive integer r. Let L be a local field, or more generally a valued field endowed with its ultrametric absolute value $|\cdot|$. A function $f : U \subseteq L^m \to L$ is said to satisfy T_r-approximation if for each $y \in U$ there is a polynomial $T_{f,y}^{<r}(x)$ of degree less than r and coefficients in L such that for each $x, y \in U$,

$$|f(x) - T_{f,y}^{<r}(x)| \leq |x - y|^r.$$

A T_r-parametrization of a set $X \subseteq L^n$ of dimension m is a finite partition of X into pieces $(X_i)_{i \in I}$ and for each $i \in I$, a subset $U_i \subseteq \mathcal{O}_L^m$ and a surjective function $f_i : U_i \to X_i$ that satisfies T_r-approximation.

The following statement is a particular case of Theorem 3.1.4.

Theorem C. *Let X be a subanalytic set included in some cartesian power of the valuation ring, and of dimension m. Then there exist integers C and N such that if L is a local field of residue characteristic $p_L \geq N$, then for each integer $r > 0$, there is a partition of $X(L)$ into Cr^m pieces such that for each piece X_i, there is a surjective function $f_i : U_i \subseteq \mathcal{O}_L^m \to X_i$ satisfying T_r-approximation on U_i.*

Observe that in the preceding theorem, we do not claim that the X_i and f_i are subanalytic, and indeed they are not in general.

Theorem C is used to deduce Theorems A and B, using an analog of the Bombieri–Pila determinant method. To be more precise, we follow closely the approach by Marmon [2010] in order to prove Theorem A.

Note also that from Theorem 3.1.3 of [Cluckers et al. 2015], we can deduce by compactness a result similar to Theorem C but for fixed r and with the number of pieces depending polynomially on the cardinality of the residue field. Such a result is however too weak to obtain a nontrivial bound in Theorem A.

The way we make Theorem C independent of the residue field is by adding algebraic Skolem functions in the residue field to the language. This enables us to work in a theory where the model-theoretic algebraic closure is equal to the definable closure. The functions involved in the parametrization are definable in such an extension of the language. Theorem C is then deduced from a T_1-parametrization Theorem 3.4.2, where the functions are required to satisfy an extra technical condition called condition $(*)$ (see Definition 3.2.1). Such a condition implies that the function (when interpreted in any local field of large enough residue characteristic) is analytic on any box contained in its domain. This allows us to deduce the T_r-parametrization result by precomposing with power functions.

A first step toward Theorem C is Theorem 2.3.1, which states that the domain of a definable (in the above sense) function that is locally 1-Lipschitz can be partitioned into finitely many definable pieces on which the function is globally 1-Lipschitz. It is similar to Theorem 2.1.7 of [Cluckers et al. 2015], but there the domain is partitioned into infinitely many pieces parametrized (definably) by the residue field. The improvement is made possible by the fact that we work in a theory with algebraic Skolem functions in the residue field.

Let us finally observe that the number of pieces of the T_r-parametrization is Cr^m, where m is the dimension. In the Archimedean setting, a similar result has recently been proven by Cluckers, Pila and Wilkie [Cluckers et al. 2020], but there the number of pieces of the T_r-parametrization is a polynomial in r of nonexplicit degree in general; in the case of $\mathbb{R}_{\mathrm{an}}$, this degree in r has meanwhile been made explicit in Theorem 2 of [Binyamini and Novikov 2019] (see also the discussion just before Lemma 3.4.4).

The paper is organized as follows. Section 2 is devoted to the fact that one can go from local to global Lipschitz continuity. In Section 3, we prove our main parametrization result. Sections 4 and 5 are devoted to applications, the first to the counting of points of bounded degree in $\mathbb{F}_q[t]$, the second to the uniform non-Archimedean Pila–Wilkie theorem.

2. Global Lipschitz continuity

For $h : D \subseteq A \times B \to C$ any function between sets and for $a \in A$, write D_a for the set $\{b \in B \mid (a, b) \in D\}$ and $h(a, \cdot)$ or h_a for the function which sends $b \in D_a$ to $h(a, b)$. We use similar notation D_a and $h(a, \cdot)$ or h_a when D is a (subset of a) Cartesian product $\prod_{i=1}^{n} A_i$ and $a \in p(D)$ for some coordinate projection $p : D \to \prod_{i \in I \subseteq \{1,\dots,n\}} A_i$.

2.1. ***Tame theories.*** We consider tame structures in the sense of [Cluckers et al. 2015, Section 2.1]. We recall their definition here.

Let $\mathcal{L}_{\mathrm{Basic}}$ be the first-order language with the sorts VF, RF and VG, and symbols for addition and a constant 0 on VF; for functions $\overline{\mathrm{ac}} : \mathrm{VF} \to \mathrm{RF}$ and $|\cdot| : \mathrm{VF} \to \mathrm{VG}$; for the order, the multiplication and a constant 0 on VG; and for a constant 0 on RF. Let $\mathcal{L}$ be any expansion of $\mathcal{L}_{\mathrm{Basic}}$. By $\mathcal{L}$-definable we mean $\varnothing$-definable in the language $\mathcal{L}$, and likewise for other languages than $\mathcal{L}$. By contrast, we use the word "definable" more flexibly in this paper and it may involve parameters from a structure. Write $\mathrm{VF}^0 = \{0\}$, $\mathrm{RF}^0 = \{0\}$, and $\mathrm{VG}^0 = \{0\}$, with a slight abuse of notation. Note that $\mathcal{L}$ may have more sorts than $\mathcal{L}_{\mathrm{Basic}}$, since it is an arbitrary expansion.

We assume that all the $\mathcal{L}$-structures we consider are models of $\mathcal{T}_{\mathrm{Basic}}$, the $\mathcal{L}_{\mathrm{Basic}}$-theory stating that VF is an abelian group, that $\mathrm{VG} = \mathrm{VG}^\times \cup \{0\}$ with $\mathrm{VG}^\times$ a (multiplicatively written) ordered abelian group, that $|\cdot| : \mathrm{VF} \to \mathrm{VG}$ is a surjective ultrametric absolute value (for groups), and that $\overline{\mathrm{ac}} : \mathrm{VF} \to \mathrm{RF}$ is surjective with $\overline{\mathrm{ac}}^{-1}(0) = \{0\}$.

Consider an $\mathcal{L}$-structure with K for the universe of the sort VF, k for RF and Γ for VG. We usually denote this structure by $(K, \mathcal{L})$.

Remark 2.1.1. Most often, K will be a valued field, k its residue field and Γ its value group (hence the sort names VF, RF and VG), although here we just require K to be a (valued) abelian group.

We define an open ball as a subset $B \subseteq K$ of the form $\{x \in K \mid |x - a| < \alpha\}$, for some $a \in K$ and $\alpha \in \Gamma^\times$, and similarly a closed ball as $\{x \in K \mid |x - a| \leq \alpha\}$.

We define $k^\times$ as $k\backslash\{0\}$. For $\xi \in k$ and $\alpha \in \Gamma$, we introduce the notation

$$A_{\xi,\gamma} = \{x \in K \mid \overline{\mathrm{ac}}(x) = \xi, |x| = \alpha\}.$$

Observe that if $\xi \in k^\times$ and $\alpha \in \Gamma^\times$, then $A_{\xi,\gamma}$ is an open ball.

We put on K the valuation topology, that is, the topology with the collection of open balls as base and the product topology on Cartesian powers of K.

For a tuple $x = (x_1, \ldots, x_n) \in K^n$, set $|x| = \max_{1 \leq i \leq n}\{|x_i|\}$.

Definition 2.1.2. Let $f : X \subseteq K^m \to K$ be a function. The function f is called 1-Lipschitz continuous (globally on X) or, in a short form, 1-Lipschitz if for all x and y in X,

$$|f(x) - f(y)| \leq |x - y|.$$

The function f is called locally 1-Lipschitz if, locally around each point of X, the function f is 1-Lipschitz continuous.

For $\gamma \in \Gamma^\times$, a function $f : X \subseteq K^n \to K$ is called γ-Lipschitz if for all x and y in X,

$$|f(x) - f(y)| \leq \gamma \cdot |x - y|.$$

Definition 2.1.3 (s-continuity). Let $F : A \to K$ be a function for some set $A \subseteq K$. We say that F is s-continuous if for each open ball $B \subseteq A$ the set $F(B)$ is either a singleton or an open ball, and there exists $\gamma = \gamma(B) \in \Gamma$ such that

$$|F(x) - F(y)| = \gamma|x - y| \text{ for all } x, y \in B. \tag{2.1.1}$$

If a function $g : U \subseteq K^n \to K$ on an open U is s-continuous in, say, the variable x_n, by which we mean that $g(a, \cdot)$ is s-continuous for each choice of $a = (x_1, \ldots, x_{n-1})$, then we write $|\partial g/\partial x_n(a, x_n)|$ for the element $\gamma \in \Gamma$ witnessing the s-continuity of $g(a, \cdot)$ locally at x_n, namely, γ is as in (2.1.1) for the function $F(\cdot) = g(a, \cdot)$, where x, y run over some ball B containing x_n and with $\{a\} \times B \subseteq U$.

Definition 2.1.4 (tame configurations). Fix integers $a \geq 0$, $b \geq 0$, a set

$$T \subseteq K \times k^a \times \Gamma^b,$$

and some $c \in K$. We say that T is in c-config if there is $\xi \in k$ such that T equals the union over $\gamma \in \Gamma$ of sets

$$(c + A_{\xi,\gamma}) \times U_\gamma$$

for some $U_\gamma \subseteq k^a \times \Gamma^b$. If moreover $\xi \neq 0$ we speak of an open c-config, and if $\xi = 0$ we speak of a graph c-config. If T is nonempty and in c-config, then ξ and the sets U_γ with $A_{\xi,\gamma}$ nonempty are uniquely determined by T and c.

We say that $T \subseteq K \times k^a \times \Gamma^b$ is in $\mathcal{L}$-tame config if there exist $s \geq 0$ and $\mathcal{L}$-definable functions

$$g : K \to k^s \quad \text{and} \quad c : k^s \to K$$

such that the range of c is finite, and, for each $\eta \in k^s$, the set

$$T \cap (g^{-1}(\eta) \times k^a \times \Gamma^b)$$

is in $c(\eta)$-config. We call c the center of T (despite not being in T in the case of open c-config).

For any $\mathcal{L}$-structure M elementarily equivalent to $(K, \mathcal{L})$ and for any language L obtained from $\mathcal{L}$ by adding some elements of M (of any sort) as constant symbols, call (M, L) a test pair for $(K, \mathcal{L})$.

Definition 2.1.5 (tameness). We say that $(K, \mathcal{L})$ is weakly tame if the following conditions hold.

(1) Each $\mathcal{L}$-definable set $T \subseteq K \times k^a \times \Gamma^b$ with $a \geq 0$, $b \geq 0$ is in $\mathcal{L}$-tame config.

(2) For any $\mathcal{L}$-definable function $F : X \subseteq K \to K$ there exist $s \geq 0$ and an $\mathcal{L}$-definable function $g : X \to k^s$ such that, for each $\eta \in k^s$, the restriction of F to $g^{-1}(\eta)$ is s-continuous.

We say that $(K, \mathcal{L})$ is tame when each test pair (M, L) for $(K, \mathcal{L})$ is weakly tame. Call an $\mathcal{L}$-theory $\mathcal{T}$ tame if for each model $\mathcal{M}$ of $\mathcal{T}$, the pair $(\mathcal{M}, \mathcal{L})$ is tame.

Recall [Cluckers et al. 2015, Corollary 2.1.11], which states that a tame theory, restricted in the sorts VF, RF, VG, is b-minimal, in the sense of [Cluckers and Loeser 2007]. In particular, one can make use of dimension theory for b-minimal structures.

2.2. *Skolem functions.* Recall that an $\mathcal{L}$-structure M has algebraic Skolem functions if for any $A \subseteq M$ every finite A-definable set $X \subseteq M^n$ admits an A-definable point. Observe that this condition is equivalent to the fact that the model-theoretic algebraic closure is equal to the definable closure. More generally, for a multisorted language, we say that a structure M has algebraic Skolem functions in the sort S if for any $A \subseteq M$ and every finite A-definable set $X \subseteq S_M^n$ there is an A-definable point, with S_M the universe for the sort S in the structure M.

We say that a theory T has algebraic Skolem functions (in the sort S), if each model has. In any case, one can algebraically Skolemize in the usual sense, that is, given a theory T in a language $\mathcal{L}$, the algebraic Skolemization of T in the sort S is the theory T^s in an expansion $\mathcal{L}^s$ of $\mathcal{L}$ obtained by adding function symbols, such that T^s has algebraic Skolem functions in the sort S and such that $(\mathcal{L}^s, T^s)$ is minimal with this property (where minimality is seen after identifying pairs with exactly the same models and definable sets); see also [Nübling 2004].

Lemma 2.2.1. *Let $\mathcal{L}$ a countable language extending $\mathcal{L}_{\text{Basic}}$ and $\mathcal{T}$ a tame $\mathcal{L}$-theory. If $\mathcal{T}$ has algebraic Skolem functions in the sort* RF, *then it also has algebraic Skolem functions in the sort* VF. *In any case, there is a countable extension $\mathcal{L}'$ of $\mathcal{L}$ by function symbols on the sort* RF *and an $\mathcal{L}'$-theory $\mathcal{T}'$ extending $\mathcal{T}$ such that $\mathcal{T}'$ has algebraic Skolem functions in the sort* RF *and hence also in the sort* VF. *Moreover, every model of $\mathcal{T}$ can be extended to an $\mathcal{L}'$-structure that is a model of $\mathcal{T}'$, and, $\mathcal{T}'$ is tame.*

Proof. Since $\mathcal{T}$ is tame, every finite definable (with parameters) set in the VF sort is in definable bijection with a definable set in the RF sort. The first statement follows: If $\mathcal{T}$ has algebraic Skolem functions in the sort RF, then also in the sort VF. In general, let us algebraically Skolemize the theory $\mathcal{T}$ in the sort RF. Denote by $\mathcal{L}'$ and $\mathcal{T}'$ the obtained language and theory. Clearly one may take $\mathcal{L}'$ to be countable. It remains to prove that $\mathcal{T}'$ is tame. One needs to check condition (1) and (2) of Definition 2.1.5. Assume that $(K, \mathcal{L}')$ is a model of $\mathcal{T}'$ and let $T \subseteq K \times k^a \times \Gamma^b$ be some $\mathcal{L}'$-definable set. Then there is an $\mathcal{L}$-definable set T_0 such that $T \subseteq T_0$ and for each $(x, \xi_0, \alpha) \in T_0$, there is ξ such that $(x, \xi, \alpha) \in T$ and $(x, \xi_0, \alpha) \in \operatorname{acl}_{\mathcal{L}}(x, \xi, \alpha)$. Indeed, an $\mathcal{L}$-formula for T_0 is made from one for T by replacing each occurrence of a new function symbol by a formula for the definable set it lands in. The fact that T_0 is in $\mathcal{L}$-tame config then implies that T is in $\mathcal{L}'$-tame config. The reasoning for (2) is similar. □

Remark 2.2.2. Let $\mathcal{L}$ be an extension of $\mathcal{L}_{\text{Basic}}$ such that any local field can be endowed with an $\mathcal{L}$-structure. Let $\mathcal{T}$ be an $\mathcal{L}$-theory such that any ultraproduct of local fields which is of residue characteristic zero is a model of $\mathcal{T}$. Consider the algebraic Skolemization $\mathcal{L}'$, $\mathcal{T}'$ in the sort RF from Lemma 2.2.1. Then one can endow every local field with an $\mathcal{L}'$-structure such that moreover any ultraproduct of such structures that is of residue characteristic zero is a model of $\mathcal{T}'$. Indeed, for each new function symbol in $\mathcal{L}' \backslash \mathcal{L}$ set the function output to be 0 if the corresponding set is empty, and to be any point in the set if nonempty. Such a choice of $\mathcal{L}'$-structure is often highly noncanonical and is not required to be compatible among field extensions.

Remark 2.2.3. Usually the Skolemization process breaks most of the model-theoretic properties of the theory. However, since we apply it only to the residue field many results such as cell decomposition are preserved. Moreover, since we add only algebraic Skolem functions in the sort RF, the situation is somehow controlled. For example, if the theory of the residue field is simple in the sense of model theory, then adding algebraic Skolem functions in the residue field preserves simplicity; see [Nübling 2004].

It is also worth noting that we will apply our results in the case where the residue field is pseudofinite, and that such fields almost always have algebraic Skolem functions; see [Beyarslan and Hrushovski 2012]. See also [Beyarslan and Chatzidakis 2017] for a more concrete characterization.

2.3. ***Lipschitz continuity.*** We can now state our first main result on Lipschitz continuity, going from local to piecewise global (with finitely many pieces).

Theorem 2.3.1. *Suppose that $(K, \mathcal{L})$ is tame with algebraic Skolem functions in the sort* RF. *Let $f : X \subseteq K^n \to K$ be an $\mathcal{L}$-definable function which is locally* 1*-Lipschitz. Then there exists a finite definable partition of X such that the restriction of f on each of the parts is* 1*-Lipschitz.*

As in [Cluckers et al. 2015], Theorem 2.3.1 is complemented by Theorem 2.3.2 about simultaneous partitions of domain and range into parts with 1-Lipschitz centers. They are proved by a joint induction on n.

Theorem 2.3.2 (Lipschitz continuous centers in domain and range). *Suppose that $(K, \mathcal{L})$ is tame with algebraic Skolem functions in the sort* RF. *Let $f : A \subseteq K^n \to K$ be an $\mathcal{L}$-definable function which is locally* 1*-Lipschitz. Then for a finite partition of A into definable parts, the following holds for each part X. There exist $s \geq 0$, a coordinate projection $p : K^n \to K^{n-1}$ and $\mathcal{L}$-definable functions*

$$g : X \to k^s, \quad c : p(X) \subseteq K^{n-1} \to K \quad \text{and} \quad d : p(X) \subseteq K^{n-1} \to K$$

such that c and d are 1*-Lipschitz, and for each $\eta \in k^s$ and w in $p(X)$, the set $g^{-1}(\eta)_w$ is in $c(w)$-config and the image of $g^{-1}(\eta)_w$ under f_w is in $d(w)$-config.*

Before proving Theorems 2.3.1 and 2.3.2, we obtain in Lemma 2.3.5 a weaker version of Theorem 2.3.2, where the centers are only required to be locally 1-Lipschitz. It will itself rely on [Cluckers et al. 2015, Theorem 2.1.8], which looks similar, but there the centers depend on auxiliary parameters.

Lemma 2.3.3. *Suppose that $(K, \mathcal{L})$ is tame with algebraic Skolem functions in the sort* RF. *Let $Y \subseteq K^n \times k^s$ be a definable set, $p : Y \to K^n$ be the canonical projection, $X = p(Y)$, and $f : X \to K$ be a definable function such that for each $\eta \in k^s$, the restriction of f to Y_η is locally* 1*-Lipschitz. Then there is a finite definable partition of X such that the restriction of f on each of the pieces is locally* 1*-Lipschitz.*

The proof of Lemma 2.3.3 is a joint induction with the following lemma.

Lemma 2.3.4. *Suppose that $(K, \mathcal{L})$ is tame with algebraic Skolem functions in the sort* RF. *Let $A \subseteq K^m$ be a definable set of dimension n. Then there is a finite definable partition of A such that for each part X, there is an injective projection $X \subseteq K^m \to K^n$ and its inverse is locally* 1*-Lipschitz.*

Proof of Lemma 2.3.4. Assume Lemma 2.3.3 holds for integers up to n. We use dimension theory for b-minimal structures. We get a finite definable partition of A such that on each piece X, there is a projection $p : X \to K^n$ which is finite-to-one. For each $w \in p(X)$, the fiber X_w is finite. By the existence of algebraic Skolem functions in the sort RF and hence also in VF by Lemma 2.2.1, each of the points of X_w is definable. By compactness, we can find a finite definable partition of X such that p is injective on each of the pieces.

By [Cluckers et al. 2015, Corollary 2.1.14], up to changing the coordinate projection we see that the inverse of p is locally 1-Lipschitz when restricted to fibers of some definable function $g : p(X) \to k^r$. By Lemma 2.3.3, we can find a finite partition of $p(X)$ such that the inverse of p is locally 1-Lipschitz on each of the parts. □

Proof of Lemma 2.3.3. We work by induction on n. If $n = 0$ there is nothing to prove. Assume now $n \geq 1$ and that Lemmas 2.3.3 and 2.3.4 hold for integers up to $n - 1$. Assume first that X is of dimension n. By dimension theory, there is at least one η such that Y_η is of dimension n. Define X' to be the union of the interior of Y_η for all such $\eta \in k^s$. The function f is locally 1-Lipschitz on X'. It remains to deal with $X'' = X \backslash X'$. By dimension theory, X'' is of dimension less than n. Assume $X'' = X$ for simplicity. By

Lemma 2.3.4, up to considering a finite definable partition of X we can assume that there is an injective coordinate projection $p : X \to K^{n-1}$ with inverse locally 1-Lipschitz. Then f is locally 1-Lipschitz if and only if $f \circ p^{-1}$ is. Now $p(X)$ with the function $f \circ p^{-1}$ satisfies the hypothesis of Lemma 2.3.3. By induction hypothesis, we have the result. □

Lemma 2.3.5. *Suppose that $(K, \mathcal{L})$ is tame with algebraic Skolem functions in the sort* RF. *Let $f : A \subseteq K^n \to K$ be an $\mathcal{L}$-definable function which is locally* 1*-Lipschitz. Then for a finite partition of A into definable parts, the following holds for each part X. There exist $s \geq 0$, a coordinate projection $p : K^n \to K^{n-1}$ and $\mathcal{L}$-definable functions*

$$g : X \to k^s, \quad c : p(X) \subseteq K^{n-1} \to K \quad \textit{and} \quad d : K^{n-1} \to K$$

such that the functions c and d are locally 1*-Lipschitz, and for each w in $p(K^n)$, the set $g^{-1}(\eta)_w$ is in $c(w)$-config and the image of $g^{-1}(\eta)_w$ under f_w is in $d(w)$-config.*

The proof uses [Cluckers et al. 2015, Theorem 2.1.8], but only a weaker version is actually needed: we only need to require the centers to be locally 1-Lipschitz.

Proof. Apply [Cluckers et al. 2015, Theorem 2.1.8] to f. Work on one of the definable pieces X of A and use notations from the application of [Cluckers et al. 2015, Theorem 2.1.8], which is similar to Theorem 2.3.2 except that the input of c and d may additionally depend on some k-variables. We now show that these additional k-variables are not needed as input for c and d. We first show (after possibly taking a finite definable partition of X) that $c(\cdot, w)$ and $d(\cdot, w)$ are constant.

Fix some $w \in p(X)$. Since the range of the w-definable function $c_w : \eta \in k^s \mapsto c(\eta, w) \in K$ does not contain an open ball, it must be finite. By tameness, there is a w-definable bijection h_w between the range of c_w and a subset of $B_w \subseteq k^{s'}$, for some $s' \in \mathbb{N}$. By the existence of algebraic Skolem functions in the sort RF, and hence also in VF by Lemma 2.2.1, each of the points of B_w is w-definable. Taking the preimage of those points by $h_w \circ c_w$ leads to a w-definable finite partition of k^s. After taking preimages by g, this itself leads to a w-definable finite partition of X_w. By compactness, we find a finite partition of X such that on each piece, the function $c(g(x), p(x))$ is independent of $g(x) \in k^s$ and can be (abusively) written $c(p(x))$. The argument for d is similar.

By Lemma 2.3.3, we can refine the partition such that the functions $c, d : p(X) \to K$ are locally 1-Lipschitz. □

Proof of Theorem 2.3.2. We proceed by induction on n. Theorem 2.3.2 for $n = 1$ is exactly Lemma 2.3.5 for $n = 1$ since the Lipschitz condition is empty in this case. Assume now that Theorems 2.3.1 and 2.3.2 hold for integers up to $n - 1$. Apply Lemma 2.3.5. On each of the definable pieces X obtained, one has a coordinate projection p and definable functions $c, d : p(X) \to K$ that are locally 1-Lipschitz. By Theorem 2.3.1 for $n - 1$, we have a finite definable partition of $p(X)$ such that c and d are 1-Lipschitz on each of the pieces. This induces a finite definable partition of X satisfying the required properties. □

Proof of Theorem 2.3.1. We work by induction on n, assuming that Theorem 2.3.2 holds for integers up to n and Theorem 2.3.1 holds for integers up to $n - 1$. For $n = 0$ there is nothing to show, and hence we assume that $n \geq 1$. Write $p : X \to K^{n-1}$ for the coordinate projection sending $x = (x_1, \dots, x_n)$ to $\hat{x} = (x_1, \dots, x_{n-1})$, and define Y as the image of X under the function $h : X \to K^n$ sending x to $(\hat{x}, f(x))$.

Up to taking a finite definable partition of X, switching the variables, by induction on the number of variables on which f depends, by Lemma 2.3.4 and Theorem 2.3.2, tameness and compactness, we may assume that the following holds:

- X is open in K^n,
- there is a definable function $g : X \to k^s$, and definable functions $c, d : p(X) \to K$,
- for each $\hat{x} \in p(X)$ and $\eta \in k^s$, $g^{-1}(\eta)_{\hat{x}}$ is in open $c(\hat{x})$-config and $h(g^{-1}(\eta))_{\hat{x}}$ is in $d(\hat{x})$-config,
- the restriction of $f(\hat{x}, \cdot)$ to $g^{-1}(\eta)_{\hat{x}}$ is s-continuous for each $\hat{x} \in p(X)$ and $\eta \in k^s$,
- the functions c and d are 1-Lipschitz,
- the function $f(\cdot, x_n)$ is 1-Lipschitz for each x_n.

We show that under these assumptions, f is 1-Lipschitz. Since d is 1-Lipschitz, we can replace f by $x \mapsto f(\hat{x}, x_n) - d(\hat{x})$ (and translate Y accordingly) in order to assume $d = 0$.

Let $x, y \in X$ and assume first that both x_n and y_n lie in an open ball $B \subseteq X_{\hat{x}}$. Then $g(x) = g(\hat{x}, y_n)$; indeed, otherwise $c(\hat{x}) \in B$, which would contradict that $g^{-1}(\eta)_{\hat{x}}$ is in open $c(\hat{x})$-config for every $\eta \in k^s$. It follows that $f(\hat{x}, \cdot)$ is s-continuous on B. Since f is locally 1-Lipschitz, the constant γ involved in the definition of s-continuity on B satisfies $\gamma \leq 1$.

Thus, using the ultrametric inequality and the assumption about $f(\cdot, y_n)$, we have

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f(\hat{x}, y_n) + f(\hat{x}, y_n) - f(y)| \\ &\leq \max\big(|f(x) - f(\hat{x}, y_n)|, |f(\hat{x}, y_n) - f(y)|\big) \\ &\leq \max\big(|x_n - y_n|, |\hat{x} - \hat{y}|\big) \\ &= |x - y|, \end{aligned}$$

which settles this case.

Suppose now that x_n and y_n do not lie in an open ball included in $X_{\hat{x}}$, and by symmetry nor in an open ball included in $X_{\hat{y}}$. This implies that

$$|x_n - c(\hat{x})| \leq |x_n - y_n| \quad \text{and} \quad |y_n - c(\hat{y})| \leq |x_n - y_n|. \tag{2.3.1}$$

By s-continuity and the fact that f is locally 1-Lipschitz, the image of a small enough open ball in $X_{\hat{x}}$ of radius α is either a point or an open ball of radius less than or equal to α. This implies that

$$|f(x) - d(\hat{x})| \leq |x_n - c(\hat{x})| \quad \text{and} \quad |f(y) - d(\hat{y})| \leq |y_n - c(\hat{y})|. \tag{2.3.2}$$

Recall that $d = 0$. Combining (2.3.1) and (2.3.2), we have by the ultrametric inequality

$$|f(x) - f(y)| \leq \max\big(|x_n - c(\hat{x})|, |y_n - c(\hat{y})|\big) \leq |x_n - y_n| \leq |x - y|,$$

which finishes the proof. $\square$

Remark 2.3.6. Let us recall that [Cluckers et al. 2010] and [Cluckers and Halupczok 2012], with related results on Lipschitz continuity on p-adic fields, are amended in Remark 2.1.16 of [Cluckers et al. 2015]. When making $d = 0$ it is important to keep c possibly nonzero in the proof of [Cluckers et al. 2015, Theorem 2.1.7] and in the above proof of Theorem 2.3.1; this was forgotten in the proofs of the corresponding results [Cluckers et al. 2010, Theorems 2.3] and [Cluckers and Halupczok 2012, Theorem 3.5], where c should also have been kept.

3. Analytic parametrizations

The goal of this section is to prove a uniform version of non-Archimedean Yomdin–Gromov parametrizations.

3.1. *T_r-approximation.*

Setting 3.1.1. We fix for the whole section one of the two following settings, of $\mathcal{T}_{\mathrm{DP}}$ or $\mathcal{T}_{\mathrm{DP}}^{\mathrm{an}}$, both of which we now introduce. Let $\mathcal{O}$ be the ring of integers of a number field. Recall that the Denef–Pas language is a three sorted language, with one sort VF for the valued field with the ring language, one sort RF for the residue field with the ring language, one sort VG for the value group with the Presburger language with an extra symbol for ∞, and function symbols $\mathrm{ord} : \mathrm{VF} \mapsto \mathrm{VG}$ for the valuation (sometimes denoted multiplicatively $|\cdot|$) and $\overline{\mathrm{ac}} : \mathrm{VF} \to \mathrm{RF}$ for an angular component map (namely a multiplicative map sending 0 to 0 and sending a unit of the valuation ring to its reduction modulo the maximal ideal). Consider the theory of henselian discretely valued fields of residue field characteristic zero in the Denef–Pas language, with constants symbols from $\mathcal{O}[[t]]$ and with t as a uniformizer of the valuation ring. This theory is tame by Theorem 6.3.7 of [Cluckers and Lipshitz 2011]. Applying Lemma 2.2.1, one obtains a new language and a new theory which we denote by $\mathcal{L}_{\mathrm{DP}}$ and $\mathcal{T}_{\mathrm{DP}}$, which thus has algebraic Skolem functions in each of the sorts.

We can also work in an analytic setting corresponding to Example 4.4(1) of [Cluckers and Lipshitz 2011], as follows. Consider the expansion of the Denef–Pas language $\mathcal{L}_{\mathrm{DP}}$ by adding function symbols for elements of

$$\mathcal{O}[[t]]\{x_1, \dots, x_n\} = \left\{ f = \sum_{I \in \mathbb{N}^n} a_I x^I \;\middle|\; a_I \in \mathcal{O}[[t]], \mathrm{ord}_t(a_I) \underset{|I| \to +\infty}{\longrightarrow} +\infty \right\}.$$

Any complete discretely valued field over $\mathcal{O}$ (namely, with a unital ring homomorphism from $\mathcal{O}$ into the valued field) can be endowed with a structure for this expansion, by interpreting the new function symbols as the corresponding power series evaluated on the unit box and put equal to zero outside the unit box. Let $\mathcal{L}_{\mathrm{DP}}^{\mathrm{an}}$ and $\mathcal{T}_{\mathrm{DP}}^{\mathrm{an}}$ be the resulting language and the theory of these models, respectively. (For a shorter and explicit axiomatization for the analytic case, see the axioms of Definition 4.3.6(i) of [Cluckers and Lipshitz 2011].)

From now on, we work in a language $\mathcal{L}$ that is either $\mathcal{L}_{\mathrm{DP}}$ or $\mathcal{L}_{\mathrm{DP}}^{\mathrm{an}}$ and in the theory $\mathcal{T}$ that is correspondingly $\mathcal{T}_{\mathrm{DP}}$ or $\mathcal{T}_{\mathrm{DP}}^{\mathrm{an}}$.

Let us summarize our theory once more: $\mathcal{T}$ is the $\mathcal{L}$-theory which is the algebraic Skolemization in the residue field sort of the theory of complete discrete valued fields, residue field of characteristic zero, with constants symbols from $\mathcal{O}[\![t]\!]$ (as a subring) and where t has valuation 1, and (in the subanalytic case), with the restricted analytic function symbols as the corresponding power series evaluated on the unit box and put equal to zero outside the unit box.

In any case, the theory $\mathcal{T}$ is tame by Theorem 6.3.7 of [Cluckers and Lipshitz 2011], and, it has algebraic Skolem functions in each sort by Lemma 2.2.1 and by Example 4.4(1) with the homothecy with factor t on the valuation ring to make the system strict instead of separated. Note that there is no need to algebraically Skolemize again when going from $\mathcal{T}_{\mathrm{DP}}$ to the larger theory $\mathcal{T}_{\mathrm{DP}}^{\mathrm{an}}$ by the elimination of valued field quantifiers from Theorem 6.3.7 of [Cluckers and Lipshitz 2011]. Definable means definable without parameters in the theory $\mathcal{T}$.

Definition 3.1.2 (T_r-approximation). Let L be any valued field. Consider a set $P \subseteq L^m$, a function $f = (f_1, \dots, f_n) : P \to \mathcal{O}_L^n$ and an integer $r > 0$. We say that f satisfies *T_r-approximation* if P is open in L^m, and, for each $y \in P$, there is an n-tuple $T_{f,y}^{<r}$ of polynomials with coefficients in $\mathcal{O}_L$ and of degree less than r that satisfies

$$|f(x) - T_{f,y}^{<r}(x)| \le |x - y|^r \quad \text{for all } x \in P.$$

We say that a family $(g_i)_{i\in I}$ of functions $g_i : P_i \to X_i \subseteq \mathcal{O}_L^n$ is a T_r-parametrization of $X = \bigcup_{i\in I} X_i$ if each g_i is surjective and satisfies T_r-approximation.

Observe that if f satisfies T_r-approximation, then the polynomials $T_{f,y}^{<r}$ are uniquely determined.

Observe also that if K is a complete valued field of characteristic zero, if f is of class $\mathscr{C}^r$ (i.e., f is r times differentiable and the r-th differential is continuous) and satisfies T_r-approximation, then $T_{f,y}^{<r}$ is just the tuple of Taylor polynomials of f at y of order r.

Notation 3.1.3. Let $\mathcal{O}$ be the ring of integers of a number field. We denote by $\mathcal{A}_{\mathcal{O}}$ the collection of all local fields of characteristic zero over $\mathcal{O}$ and by $\mathcal{B}_{\mathcal{O}}$ those of positive characteristic, and set $\mathscr{C}_{\mathcal{O}} = \mathcal{A}_{\mathcal{O}} \cup \mathcal{B}_{\mathcal{O}}$. (By a local field L over $\mathcal{O}$ we mean a non-Archimedean locally compact field, i.e., a finite field extension of $\mathbb{Q}_p$ or of $\mathbb{F}_p((t))$ for a prime p, allowing a unital homomorphism $\mathcal{O} \to L$.) If $L \in \mathscr{C}_{\mathcal{O}}$, we denote by ord its valuation (normalized such that $\mathrm{ord}(L^\times) = \mathbb{Z}$), $\mathcal{O}_L$ its valuation ring, $\mathcal{M}_L$ its maximal ideal, $\varpi_L \in \mathcal{M}_L$ a fixed choice of uniformizer, k_L its residue field, q_L the cardinality of k_L and p_L the characteristic of k_L. If $N \in \mathbb{N}$, we define $\mathcal{A}_{\mathcal{O},N}$ (resp. $\mathcal{B}_{\mathcal{O},N}$, $\mathscr{C}_{\mathcal{O},N}$) to be the set of $L \in \mathcal{A}_{\mathcal{O}}$ (resp. $L \in \mathcal{B}_{\mathcal{O}}$, $L \in \mathscr{C}_{\mathcal{O}}$) such that $p_L \ge N$. By Remark 2.2.2, we can consider $L \in \mathscr{C}_{\mathcal{O}}$ as an $\mathcal{L}$-structure, and any nonprincipal ultraproduct of residue characteristic zero of such local fields is a model of $\mathcal{T}$.

A family $(X_y)_{y\in Y}$ of sets X_y indexed by $y \in Y$ is called a definable family if the total set $\mathcal{X} := \{(x, y) \mid x \in X_y,\ y \in Y\}$ (and hence also Y) is a definable set. Likewise, a family of functions is called a definable family if the family of graphs is a definable family. We use notations like $\mathcal{O}_{\mathrm{VF}}$ for the definable set which in any model K is the valuation ring $\mathcal{O}_K$, and similarly $\mathcal{M}_{\mathrm{VF}}$ for the maximal ideal, and so on. For a definable set X and a structure L, we write $X(L)$ for the L-points on X, and for a definable function $f : X \to Y$, we write f_L for the corresponding function $X(L) \to Y(L)$.[1]

The main goal of this section is to prove the next two theorems on the existence of T_r-parametrizations with rather few maps, in terms of r. Even the mere finiteness of the parametrizing maps is new, as compared to [Cluckers et al. 2015] where "residue many" maps were allowed, but we even get an upper bound which is polynomial in r. This finiteness is crucial for Theorem A, and, useful for Theorem B, where it makes the exponent α of q_L independent of X. Recall from Setting 3.1.1 that we work in a theory with algebraic Skolem functions.

Theorem 3.1.4 (uniform T_r-approximation in local fields). *Let $n \geq 0$, $m \geq 0$ be integers and $X = (X_y)_{y \in Y}$ a definable family of subsets $X_y \subseteq \mathcal{O}_{\mathrm{VF}}^n$, for y running over a definable set Y. Suppose that X_y has dimension m for each $y \in Y$ (and in each model of $\mathcal{T}$). Then there exist integers $c > 0$ and $M > 0$ such that for each $L \in \mathscr{C}_{\mathcal{O},M}$ and for each integer $r > 0$, there are a finite set $I_{r,q}$ of cardinality cr^m and an R_r-definable family $g = (g_{y,i})_{(y,i) \in Y(L) \times I_r}$ of (R_r, y)-definable functions*

$$g_{y,i} : P_{y,i} \to X_y(L)$$

with $P_{y,i} \subseteq \mathcal{O}_L^m$ such that for each $y \in Y(L)$, the family $(g_{y,i})_{i \in I_{r,q}}$ forms a T_r-parametrization of $X_y(L)$ and $R_r \subset \mathcal{O}_L^\times$ is a set of lifts of representatives for the r-th powers in $\mathbb{F}_{q_L}^\times$.

Note that Theorem C in the introduction is a particular case and a less precise version of Theorem 3.1.4.

The following result is uniform in all models K of $\mathcal{T}$. Note that $\mathcal{T}$ requires in particular the residue field to have characteristic zero, and the value group to be elementarily equivalent to $\mathbb{Z}$.

Theorem 3.1.5 (uniform T_r-approximation for models of $\mathcal{T}$). *Let $n \geq 0$, $m \geq 0$ be integers and let $X = (X_y)_{y \in Y}$ be a definable family of subsets $X_y \subseteq \mathcal{O}_{\mathrm{VF}}^n$, for y running over a definable set Y. Suppose that X_y has dimension m for each $y \in Y$ and each model of $\mathcal{T}$. Then there exists an integer $c > 0$ such that for each model K of $\mathcal{T}$ and for each integer $r > 0$ such that the r-th powers in the residue field have a finite number $b_r = b_r(K)$ of cosets, there are a finite set I_r of cardinality $c(b_r r)^m$ and an R_r-definable family $g = (g_{y,i})_{(y,i) \in Y(K) \times I_r}$ of (R_r, y)-definable functions*

$$g_{y,i} : P_{y,i} \to X_y(K)$$

with $P_{y,i} \subseteq \mathcal{O}_K^m$ such that for each $y \in Y(K)$, the family $(g_{y,i})_{i \in I_r}$ forms a T_r-parametrization of $X_y(K)$ and $R_r \subset \mathcal{O}_K^\times$ is a set of lifts of representatives for the r-th powers in $k^\times$.

Remark 3.1.6. Observe that even if Theorems 3.1.4 and 3.1.5 are very similar, one cannot deduce the first from the second by compactness. The reason is the quantification over r in the statement. They will, however, both be deduced from the upcoming Theorem 3.4.2, which is a T_1-parametrization theorem

[1] When we interpret definable sets or functions into local fields L (or, more generally, $\mathcal{L}$-structures that are not models of our theory $\mathcal{T}$), we implicitly assume that we have chosen some formula φ that defines the set and consider $\varphi(L)$. This set $\varphi(L)$ may of course change with a different choice of formula φ for small values of the residue field characteristic of L, but this is not a problem by Remark 2.2.2, and since we are interested only in the case of large residue field characteristic.

with an extra technical condition. It will allow us to define a T_r-parametrization by precomposing by power functions. Furthermore, note that in Theorem 3.1.4, the factor b_r for the index of r-th powers in the residue field is not needed; this is because of an additional trick using a property true in finite fields.

Remark 3.1.7. For most of the section, we could in fact work in a slightly more general setting (up to imposing some additional requirements for Theorem 3.1.4). Using resplendent relative quantifier elimination as in [Rideau 2017], we can add arbitrary constant symbols and allow an arbitrary residual extension (and an arbitrary extension on the value group) of the language and the theory before applying the algebraic Skolemization in the residue field sort. In particular, Theorem 3.1.5 holds in this more general setting. If the extended language and theory still have the property that any local field can be equipped with a structure for the extended language such that, moreover, any ultraproduct of such equipped local fields which is of residue characteristic zero is a model of the extended theory, then also Theorem 3.1.4 would go through.

Remark 3.1.8. The condition that the value group be a Presburger group can probably be relaxed to any value group in which the index v_r of the subgroup of r-multiples is finite, by replacing $c(b_r r)^m$ by $c(b_r v_r)^m$ for the cardinality of I_r and taking $R_r \cup V_r$ instead of R_r, V_r a set of lifts of representatives for the r-multiples in the value group.

Note that extending Theorem 3.1.5 and its proof to mixed characteristic henselian valued fields may be possible too, with the adequate adaptations. For example, when going from local to piecewise Lipschitz continuity, the Lipschitz constant should be allowed to grow. (Indeed, look at the function $x \mapsto x^p$ on the valuation ring of $\mathbb{C}_p$.)

Before starting the proofs of Theorems 3.1.4 and 3.1.5, we need a few more definitions.

Definition 3.1.9 (cell with center). Consider an integer $n \geq 0$. For nonempty definable sets Y and $X \subseteq Y \times \mathrm{VF}^n$, the set X is called a cell over Y with center $(c_i)_{i=1,\dots,n}$ if it is of the form

$$\big\{(y, x) \in Y \times \mathrm{VF}^n \mid \big(y, \big(\overline{\mathrm{ac}}(x_i - c_i(x_{<i})), |x_i - c_i(x_{<i})|\big)_{i=1}^n\big) \in G\big\},$$

for some definable set $G \subseteq Y \times \mathrm{RF}^n \times \mathrm{VG}^n$ and some definable functions and $c_i : Y \times \mathrm{VF}^{i-1} \to \mathrm{VF}$, where $x_{<i} = (y, x_1, \dots, x_{i-1})$. If moreover G is a subset of $Y \times (\mathrm{RF}^\times)^n \times (\mathrm{VG}^\times)^n$, where $(\mathrm{VG}^\times)^0 = \{0\}$, then X is called an open cell over Y (with center $(c_i)_{i=1,\dots,n}$).

We next give a special name to cells over Y whose center equals 0.

Definition 3.1.10 (cell around zero). We say that a nonempty set $X \subseteq Y \times \mathrm{VF}^n$ is a cell around zero (over Y) if it is of the form

$$X = \big\{(y, x) = (y, x_1, \dots, x_n) \in Y \times \mathrm{VF}^n \mid \big(y, (\overline{\mathrm{ac}}(x_i), |x_i|)_{i=1}^n\big) \in G\big\}$$

for some definable set $G \subseteq Y \times \mathrm{RF}^n \times \mathrm{VG}^n$. Similarly, one can call a set X a cell around zero (over Y) for $X \subset L^n$ for some valued field L with an angular component map, if it is of the corresponding form.

Definition 3.1.11 (associated cell around zero). Let X be a cell over Y with center, with notation from Definition 3.1.9. The cell around zero associated to X is by definition the cell $X^{(0)}$ obtained by forgetting the centers, namely

$$X^{(0)} = \{(y, x) \in Y \times \mathrm{VF}^n \mid y \in Y,\ \overline{\mathrm{ac}}(x_i) = \xi_i(y),\ (y, (|x_i|)_i) \in G\}$$

with associated bijection $\theta_X : X \to X^{(0)}$ sending (y, x) to $(y, (x_i - c_i(x_{<i}))_i)$. For a definable map $f : X \to Z$ there is the natural corresponding function $f^{(0)} = f \circ \theta_X^{-1}$ from $X^{(0)}$ to Z.

Definition 3.1.12 (associated box). Let K be a valued field. By a box $B \subset K^n$ we mean a product of open balls in K. Let $B = \prod_{1\le i\le n} B(a_i, r_i) \subseteq K^n$ be a box, with open balls

$$B(a_i, r_i) = \{x \in K \mid |x - a_i| < r_i\},$$

with $a_i \in K$ and nonzero $r_i \in \Gamma_K$. The box associated to B is the box $B_{\mathrm{as}} \subseteq K^{\mathrm{alg}}$ defined by

$$B_{\mathrm{as}} = \{x \in (K^{\mathrm{alg}})^n \mid |x - a_i| < r_i\},$$

where K^{alg} is an algebraic closure of K, endowed with the canonical extension of the valuation of K.

We now define the term language. This is an expansion $\mathcal{L}^*$ of $\mathcal{L}$, by joining division and witnesses for henselian zeros and roots.

Definition 3.1.13. Let $\mathcal{L}^*$ be the expansion of $\mathcal{L} \cup \{^{-1}\}$ obtained by joining to $\mathcal{L} \cup \{^{-1}\}$ function symbols h_m and root_m for integers $m > 1$. The h_m are interpreted on a henselian valued field K of equicharacteristic zero and residue field k as the functions

$$h_m : K^{m+1} \times k \to K$$

sending $(a_0, \dots, a_m, \xi)$ to the unique y satisfying $\mathrm{ord}(y) = 0$, $\overline{\mathrm{ac}}(y) \equiv \xi \mod \mathcal{M}_K$, and $\sum_{i=0}^m a_i y^i = 0$, whenever ξ is a unit, $\mathrm{ord}(a_i) \ge 0$, $\sum_{i=0}^m a_i \xi^i \equiv 0 \mod \mathcal{M}_K$, and

$$f'(\xi) \not\equiv 0 \mod \mathcal{M}_K$$

with f' the derivative of f, and to 0 otherwise. Likewise, root_m is the function $K \times k \to K$ sending (x, ξ) to the unique y with $y^m = x$ and $\overline{\mathrm{ac}}(y) = \xi$ if there is such y, and to 0 otherwise.

Proposition 3.1.14 (term structure of definable functions). *Every* VF*-valued definable function is piecewise given by a term. More precisely, given a definable set X and a definable function $f : X \to$* VF, *there exists a finite partition of X into definable parts and for each part A an $\mathcal{L}^*$-term t such that*

$$t(x) = f(x) \quad \textit{for all } x \in A.$$

Proof. By Theorem 7.5 of [Cluckers et al. 2006] there exists a definable function $g : X \to \mathrm{RF}^m$ for some $m \ge 0$ and an $\mathcal{L}^*$-term t_0 such that

$$t_0(x, g(x)) = f(x).$$

Since the terms h_n (the henselian witnesses) and root_n (the root functions) involve at most a finite choice in the residue field, one can reduce to the case that g has finite image. The fibers of g can then be taken as part of the partition to end the proof. □

3.2. *Condition* (∗). We now introduce a technical condition, named (∗), that will be used in Section 3.3 to show a strong form of analyticity of definable functions, named global analyticity in Definition 3.3.1.

Definition 3.2.1 (condition (∗)). We first define condition (∗) for $\mathcal{L}^*$-terms, inductively on the complexity of terms. Consider a definable set $X \subseteq \mathrm{VF}^m$ and let x run over X.

We say that a VF-valued $\mathcal{L}^*$-term $t(x)$ satisfies condition (∗) on X if the following holds.

If $t(x)$ is a term of complexity 0 (i.e., a constant or a variable), then it satisfies condition (∗) on X.

Suppose now that the term t is of the form $t_1 + t_2$, $t_1 \cdot t_2$, t_0^{-1}, $h_n(t_0, \dots, t_n; t_{-1})$, $\mathrm{root}_n(t_0; t_{-1})$ for some $n > 0$, or $\underline{f}(t_1, \dots, t_n)$, with $\underline{f}$ one of the analytic functions of the language. In the first two cases, we just require that t_1 and t_2 satisfy condition (∗) on X. In the remaining four cases, we require that $t_0, \dots, t_n$ satisfy condition (∗) on X and moreover that for any box $B \subseteq X$, the functions t_{-1} and $\overline{\mathrm{ac}}(t_0), \dots, \overline{\mathrm{ac}}(t_n)$, $\mathrm{ord}(t_0), \dots, \mathrm{ord}(t_n)$ are constant on B.

We finally say that an $\mathcal{L}$-definable function $f : X \subseteq \mathrm{VF}^m \to \mathrm{VF}^{m'}$ for $m' > 0$ satisfies condition (∗) on X if there is a tuple t of $\mathcal{L}^*$-terms $t_i(x)$ satisfying condition (∗) on X and such that $f(x) = t(x)$ for $x \in X$.

The following lemma ensures existence of functions satisfying condition (∗).

Lemma 3.2.2. *Let $f : X \subseteq Y \times \mathrm{VF}^m \to \mathrm{VF}^{m'}$ be a definable function for some m and m'. Then there is a finite partition of X into some open cells A over Y with center $(c_i)_{i=1,\dots,m}$ and a set B such that B_y is of dimension less than m for each $y \in Y$, such that the function*

$$(A^{(0)})_y \to \mathrm{VF}^{m'} : x \mapsto f^{(0)}(y, x)$$

satisfies condition (∗) on $(A^{(0)})_y$ for each y, with notation from Definition 3.1.11.

Proof. We proceed by induction on m. By Proposition 3.1.14 for f we may suppose that f is given by a tuple $t(x)$ of $\mathcal{L}^*$-terms. Let $h : X \to \mathrm{RF}^s \times \Gamma^{s'}$ be the definable function created from t such that h has a component function of the form t' for each RF-valued subterm t' of t and also of the forms $\mathrm{ord}(t'')$ and $\overline{\mathrm{ac}}(t'')$ for each VF-valued subterm t'' of t. The proposition requires us to find a finite partition of X into cells over Y such that for each open cell A over Y, the map $(f_{|A})^{(0)}(y, \cdot)$ has condition (∗) on $A_y^{(0)}$, with notation from Definition 3.1.11. Now apply the cell decomposition theorem adapted to h and work on one of the open pieces A. Thus, A is an open cell over Y with some center $(c_i)_{i=1,\dots,m}$ adapted to h, namely, there are definable functions $c_i : A^i \subseteq \mathrm{VF}^i \to \mathrm{VF}$ for $i = 0, \dots, m-1$ such that $h^{(0)}$ is constant on each box contained in $c^{-1}(A)$, which is moreover an open cell around zero, where

$$c : x \in \mathrm{VF}^m \mapsto (x_1 + c_0, x_2 + c_1(x), \dots, x_m + c_{m-1}(x)),$$

with notation from Definition 3.1.11. Note that $c = \theta_A^{-1}$ and $c^{-1}(A) = A^{(0)}$ in that notation. □

3.3. ***Global analyticity.*** To more easily speak of analyticity in this section, we work with complete discretely valued fields (a meaning of analyticity exists for all models of $\mathcal{T}$ by [Cluckers and Lipshitz 2011]).

Definition 3.3.1 (globally analytic map). Let K be a complete discretely valued field. Let $X \subseteq K^m$ be a set and $f : X \to K^n$ a function. We say that f is globally analytic on X if for each box $B \subseteq X$, the restriction of f to B is given by a tuple of power series with coefficients in K (say, taken around some $a \in B$), which converges on the associated box B_{as}.[2]

The following proposition is the reason why we introduced condition $(*)$. Observe that it applies also to local fields, and thus not only to models of our theory $\mathcal{T}$.

Proposition 3.3.2 (analyticity, [Cluckers and Lipshitz 2011, Lemma 6.3.15]). *Let f be a definable function satisfying condition $(*)$ on some definable set X. Then there is some $M > 0$ such that for L which is either a local field with residue field cardinality at least M, or a model of $\mathcal{T}$ which is moreover a complete discretely valued field, the following holds. For any box $B \subseteq X(L)$ and $b \in B$, there is a power series g centered at b and converging on B_{as} such that f is equal to g on B. Moreover, M can be taken uniformly in definable families of definable functions.*

Proof. We recall the strategy of the proof of [Cluckers and Lipshitz 2011, Lemma 6.3.15]. One works by induction on the complexity of the $\mathcal{L}^*$-term corresponding to the definition of condition $(*)$, using compositions of power series as in Remark 4.5.2 of [Cluckers and Lipshitz 2011]. The only nontrivial cases are t_0^{-1}, $h_n(t_0, \dots, t_n; t_{-1})$, $\mathrm{root}_n(t_0; t_{-1})$, and $\underline{f}(t_1, \dots, t_n)$ for some restricted analytic function $\underline{f}$ from the language. If L is a model of $\mathcal{T}$, we may assume by the definition of condition $(*)$ that the terms t_i satisfy condition $(*)$ on X and that t_{-1} and $\overline{\mathrm{ac}}(t_0), \dots, \overline{\mathrm{ac}}(t_n), \mathrm{ord}(t_0), \dots, \mathrm{ord}(t_n)$ are constant on B. In the local field case, by compactness there is some $M > 0$ such that if the residue field of L is of cardinality at least M, the functions t_{-1} and $\overline{\mathrm{ac}}(t_0), \dots, \overline{\mathrm{ac}}(t_n), \mathrm{ord}(t_0), \dots, \mathrm{ord}(t_n)$ are constant on any box B contained in $X(L)$. One finishes exactly as in the proof of [Cluckers and Lipshitz 2011, Lemma 6.3.11], where for the case $\underline{f}(t_0, \dots, t_n)$, with $\underline{f}$ one of the analytic functions of the language, condition $(*)$ ensures that either the function f is interpreted as the zero function on a box B or the image of the box B by $(t_1, \dots, t_n)$ is strictly contained in the unit box, whence so is the image of B_{as}, ensuring convergence of f on it, and giving analyticity of $\underline{f}(t_0, \dots, t_n)$ on B_{as}. □

3.4. ***Strong T_r-approximation.*** We can now state a stronger notion of T_r-approximation, for definable functions. The strong T_1-approximation will be key for the proofs of Theorems 3.1.4 and 3.1.5. Strong T_r-approximation for $r > 1$ is not needed in this paper, but we include its definition for the sake of completeness.

Definition 3.4.1 (strong T_r-approximation). Let $P \subseteq \mathrm{VF}^m$ be definable, $f = (f_1, \dots, f_n) : P \to \mathrm{VF}^n$ a definable function, and $r > 0$ an integer.

[2] Here, converging on B_{as} means that the partial sums obtained by evaluating at any element of B_{as} form a Cauchy sequence (the limits actually lie inside K^{alg} by [Cluckers and Lipshitz 2011]).

(1) We say that f satisfies strong T_r-approximation if P is an open cell around zero, f satisfies condition $(*)$ on P and, for each model L of $\mathcal{T}$, the function f_L satisfies T_r-approximation and moreover for each box $B \subseteq P(L)$, the $\mathcal{L}^*$-term associated to f satisfies T_r-approximation on B_{as}.

(2) A family $f_i : P_i \to X$ for $i \in I$ of definable functions is called a (strong) T_r-parametrization of $X \subseteq \mathrm{VF}^n$ if each f_i is a (strong) T_r-approximation and

$$\bigcup_{i\in I} f_i(P_i) = X.$$

The fact that P is an open cell around zero in Definition 3.4.1 is particularly handy since it enables an easy description of the maximal boxes contained in P, which combines well with condition $(*)$ and for composing with power maps. Global analyticity in complete models as given in Section 3.3, together with a calculation on the coefficients of the occurring power series, will then complete the proofs of the parametrization Theorems 3.1.4 and 3.1.5.

Theorem 3.4.2 (strong T_1-parametrization). *Let $n \geq 0$, $m \geq 0$ be integers and let $X = (X_y)_{y\in Y}$ be a definable family of subsets $X_y \subseteq \mathcal{O}_{\mathrm{VF}}^n$ for y running over a definable set Y. Suppose that X_y has dimension m for each $y \in Y$. Then there exist a finite set I and a definable family $g = (g_{y,i})_{(y,i)\in Y\times I}$ of definable functions*

$$g_{y,i} : P_{y,i} \to X_y$$

such that $P_{y,i} \subseteq \mathcal{O}_{\mathrm{VF}}^m$ and for each y, $(g_{y,i})_{i\in I}$ forms a strong T_1-parametrization of X_y.

Proof. We work by induction on m. We repeatedly throw away pieces of lower dimension and treat them by induction, working uniformly in y. We also successively consider finite definable partitions of X without renaming. By Lemma 2.3.4, up to taking a finite definable partition of X, we can find a locally 1-Lipschitz surjective function $f_y : P_y \subseteq \mathrm{VF}^m \to X_y$, with P_y open for each $y \in Y$. By Theorem 2.3.1, we can further assume that f_y is globally 1-Lipschitz on P_y, or equivalently, that f_y satisfies T_1-approximation on P_y. By Proposition 3.1.14 we may moreover suppose that the component functions of f are given by $\mathcal{L}^*$-terms. We still need to improve f and P in order for the f_y to satisfy strong T_1-approximation, in particular, condition $(*)$, T_1-approximation on associated boxes of boxes in its domain, and that P_y is an open cell around zero.

First we ensure, as an auxiliary step, that the first partial derivatives of the f_y are bounded by 1 on the associated box of any box in its domain P_y, by passing to an algebraic closure $\mathrm{VF}^{\mathrm{alg}}$ of VF with the natural $\mathcal{L}$ and $\mathcal{L}^*$ structures. This passage to $\mathrm{VF}^{\mathrm{alg}}$ preserves well properties of quantifier-free formulas and of terms by results from [Cluckers and Lipshitz 2011; 2017] for the involved analytic structures on VF and on $\mathrm{VF}^{\mathrm{alg}}$. This step is done by switching again the order of coordinates as in the proof of Lemma 2.3.4 where necessary. Since it is completely similar to the corresponding part of the proof of [Cluckers et al. 2015, Theorem 3.1.3], we skip the details.

Finally we show that we can ensure all remaining properties, using induction. Apply Lemma 3.2.2, uniformly in y, to obtain a partition of $P = (P_y)_y$ into open cells $A = (A_y)_y$ over Y with center $(c_i)_{i=1}^m$

and an associated bijection θ_A in the notation of Definition 3.1.11, while neglecting a definable subset B of P where B_y is of dimension less than m. By induction on m, we may apply Theorem 3.4.2 (for the value $m-1$) to the graph of $(c_i)_{i=1}^m$ to find a strong T_1-parametrization for this graph. One obtains the required parametrization of X by composing the parametrization of the graph of $(c_i)_{i=1}^m$ with θ_A^{-1} and f. Indeed, one first concludes as in the proof of Lemma 3.2.2 that property $(*)$ is satisfied for this composition and that the domain is an open cell around zero. Secondly, the composition of 1-Lipschitz functions is 1-Lipschitz, and the first-order partial derivatives are bounded by 1 on associated boxes of its domain. Finally, the condition of T_1-approximation on each associated box follows from Proposition 3.3.2 and [Cluckers et al. 2015, Corollary 3.2.12], since the derivative is bounded by 1 on associated boxes of its domain. □

The whole purpose of requiring the domains of strong T_1-parametrizations to be cells around zero is to deduce existence of T_r-parametrizations from strong T_1-parametrizations by precomposing with power functions. This is enabled by the next two lemmas.

Lemma 3.4.3. *Let f be a definable function on $X \subset \mathrm{VF}$ satisfying strong T_1-approximation. Then there is some $M > 0$ such that for L either a model of $\mathcal{T}$ which is a complete discretely valued field, or a local field with residue field cardinality at least M, the following holds for any integer $r > 0$ and with p_r being the r-power map sending x in L to x^r. For any open ball $B = b(1+\mathcal{M}_L) \subseteq L$ with $B \subset X$, and for any ball $D \subseteq L$ satisfying $p_r(D) \subseteq B$, the function*

$$f_r := f_L \circ p_r$$

satisfies T_r-approximation on D. Moreover, f_r can be developed around any point $b' \in D$ as a power series which is converging on D_{as} and whose coefficients c_i satisfy

$$|c_i| \le |b'|^{r-i} \quad \text{for all } i > 0.$$

Proof. Observe first that since the choice of $b \in B$ is arbitrary, it suffices to show the lemma for $b' \in D$ with $b'^r = b$. Since f satisfies condition $(*)$, there is a converging power series $\sum_{i\in\mathbb{N}} a_i(x-b)^i$ as given by Proposition 3.3.2. Since $x \mapsto \sum_{i\in\mathbb{N}} a_i(x-b)^i$ satisfies T_1-approximation on B_{as}, we have

$$\left|\sum_{i\ge 1} a_i(x-b)^i\right| < |b|$$

for all $x \in B_{\mathrm{as}}$. By the relation between the Gauss norm and the supremum norm on B_{as}, we then have

$$|a_i| \le |b|^{1-i} \tag{3.4.1}$$

for all $i \ge 1$. Fix $b' \in D$ with $b'^r = b$. Since f is given by a power series on B, by composition we can develop $f_r = \sum_{k\ge 0} c_k(x-b')^k$ as a power series around b'. Using multinomial development, we find that for $k \ge 1$,

$$|c_k| \le \max_{i\ge 1}\{|a_i| \cdot |b'|^{ri-k}\}.$$

Note that we could also get an explicit expression for c_k using the chain rule for Hasse derivatives.

Combining with equation (3.4.1) yields

$$|c_k| \le |b'|^{r-k}.$$

In particular, we have $|c_k| \le 1$ for $k \le r$ and for any $x \in D$,

$$|f_r(x) - T^{<r}_{f_r,b'}(x)| = \left|\sum_{k\ge r} c_k(x-b')^k\right| \le |x-b'|^r,$$

which concludes the proof. □

We now formulate a multidimensional version of Lemma 3.4.3. To do so we introduce the following notations. For a tuple $i = (i_1, \dots, i_m) \in \mathbb{N}^m$ and $x = (x_1, \dots, x_m) \in L^m$, recall that x^i is $\prod_{1\le k\le m} x_k^{i_k}$ and $|i| = i_1 + \cdots + i_m$. Also define $|x|_{\min,i}$ to be

$$\min_{1\le j\le m,\ i_j>0} \{|x_j|\}.$$

The idea is also to precompose with the r-th power to achieve the T_r-property on boxes. A naive approach to estimate the coefficients of the composite function, using the maximum modulus principle on the associated box, would lead to a bound for the $i \in \mathbb{N}^m$ coefficient of $|b|^r|b^i|^{-1}$. This however is not optimal and not enough for our needs. We improve it, working one variable at a time. The same idea (of composing with r-th power maps while controlling how many pieces are needed) is used in the real case in [Cluckers et al. 2020], but in our situation we get sharper control on the number of pieces in terms of r, resembling the sharper control of [Binyamini and Novikov 2019]. The difficulty for the corresponding control in [Cluckers et al. 2020] is that the cells in the o-minimal case have cell walls which also need to get small derivatives, and, composing with powers maps changes these cell walls. In our situation, there are no cell walls which can be considered as an advantage. On the other side, the absence of cell walls, and more generally of convexity arguments, has been a challenge in the non-Archimedean case that we have overcome by working with T_r-maps here and in [Cluckers et al. 2015].

Lemma 3.4.4. *Let f be a definable function on $X \subset \mathrm{VF}^m$ satisfying strong T_1-approximation. Then there is some $M > 0$ such that for L either a model of $\mathcal{T}$ which is a complete discretely valued field, or a local field with residue field cardinality at least M, the following holds for any integer $r > 0$.*

Let $b = (b_1, \dots, b_m)$ be in L^m and suppose that $B = \prod_i b_i(1+\mathcal{M}_L) \subseteq L^m$ is a subset of $X(L)$. For any $d = (d_1, \dots, d_m)$ in L^m, write $p_{r,d}$ for the function $(x_1, \dots, x_m) \mapsto (d_1x_1^r, \dots, d_mx_m^r)$. Then for any box $D \subseteq L^m$ such that $p_{r,d}(D) \subseteq B$, the function

$$f_{r,d} := f_L \circ p_{r,d}$$

satisfies T_r-approximation on D. Moreover, $f_{r,d}$ can be developed around any point $b' \in D$ as a power series converging on D_{as} with coefficients c_k satisfying

$$|c_k| \le |b'|^r_{\min,k}|b'^k|^{-1} \quad \textit{for all } k \in \mathbb{N}^m \setminus \{0\}.$$

Proof. Up to rescaling, we can assume $d_1 = \cdots = d_m = 1$. As in the proof of Lemma 3.4.3, we can fix $b \in B$, $b' \in D$ such that $b'^r = b$ and develop f as a power series $\sum_{i\in\mathbb{N}^m} a_i(x-b)^i$ that converges on B_{as}. Fix $\hat{x}_1 \in \hat{b}(1+\mathcal{M}_L)^{m-1}_{\mathrm{as}}$ and consider the function

$$f_{\hat{x}_1} : b_1(1+\mathcal{M}_L)_{\mathrm{as}} \to L, \quad x_1 \mapsto f(x_1, \hat{x}_1).$$

It is given by a power series $\sum_{i_1\in\mathbb{N}} a_{i_1}(\hat{x}_1)(x_1-b_1)^{i_1}$ around b_1 that converges on $b_1(1+\mathcal{M}_L)_{\mathrm{as}}$.

By the T_1-property for f on B_{as}, we have that for any $x_1 \in b_1(1+\mathcal{M}_L)_{\mathrm{as}}$,

$$|f_{\hat{x}_1}(x_1) - f_{\hat{x}_1}(b_1)| = |f(x_1,\hat{x}_1) - f(b_1,\hat{x}_1)| \le |x_1 - b_1| \le |b_1|.$$

Hence by the relation between the Gauss norm and the supremum norm on $b_1(1+\mathcal{M}_L)_{\mathrm{as}}$, for each $i_1 > 0$ we have

$$|a_{i_1}(\hat{x}_1)| \le |b_1|^{1-i_1}.$$

Now view $a_{i_1}(\hat{x}_1)$ as a function of $\hat{x}_1 \in \hat{b}(1+\mathcal{M}_L)^{m-1}_{\mathrm{as}}$, and by using again the relation between Gauss norm and sup norm, we find that for each $i \in \mathbb{N}$ such that $i_1 > 0$,

$$|a_i| \le |b_1|^{1-i_1} \cdot \left|\hat{b}^{(i_2,\dots,i_m)}\right|^{-1} = |b_1||b^i|^{-1}.$$

By switching the numbering of the coordinates, we get that for each $i \in \mathbb{N}^m\backslash\{0\}$,

$$|a_i| \le |b|_{\min,i}|b^i|^{-1}.$$

The end of the proof is now similar to that of Lemma 3.4.3. Indeed, we develop $f_{r,d} = f \circ p_{r,d}$ into a power series around b', denoted by $\sum_{c_k\in\mathbb{N}^m} c_k(x-b')$. Then by multinomial development and using the bound for a_i we find that for $k \in \mathbb{N}^m\backslash\{0\}$,

$$|c_k| \le |b'|^r_{\min,k}|b'^k|^{-1}.$$

It is now a direct consequence of this bound that $|c_k| \le 1$ for $k \in \mathbb{N}^m\backslash\{0\}$ with $|k| < r$.

Now fix $x \in D$ and $k \in \mathbb{N}^m\backslash\{0\}$ with $|k| \ge r$. Choose some $\underline{r} \in \mathbb{N}^m$ such that $|\underline{r}| = r$ and $\underline{r}_j \le k_j$ for $j = 1, \dots, m$. We have

$$\begin{aligned}|c_k(x-b')^k| &\le |b'|^r_{\min,k}|b'^k|^{-1}|(x-b')^k| \\ &\le |b'|^r_{\min,k}|b'^k|^{-1}|(x-b')^{k-\underline{r}}||x-b'|^r \\ &\le |b'^{\underline{r}}||b'^k|^{-1}|(x-b')^{k-\underline{r}}||x-b'|^r \\ &\le |x-b'|^r.\end{aligned}$$

Hence $f_{r,d}$ satisfies T_r-approximation on D. □

Proof of Theorem 3.1.4. First apply Theorem 3.4.2 to X to get a finite set I and a family $g = (g_{y,i})_{(y,i)\in Y\times I}$ of definable functions

$$g_{y,i} : P_{y,i} \to X_y$$

such that $P_{y,i} \subseteq \mathcal{O}^m_{\mathrm{VF}}$ and for each y, $(g_{y,i})_{i\in I}$ forms a strong T_1-parametrization of X_y.

By Proposition 3.3.2, we find $M \in \mathbb{N}$ such that for any $L \in \mathscr{C}_{\mathcal{O},M}$, any $y \in Y(L)$, any box $B \subseteq P_{y,i}(L)$ and any $b \in B$, there is a power series centered at b, converging on B_{as} and equal on B_{as} to g_{as}. Fix such an L and write q for q_L.

Observe that it is enough to prove the theorem for r prime to q. Indeed, a T_{r+1}-parametrization is also a T_r-parametrization. Hence, up to enlarging the constant, if r is not prime to q one can apply the theorem with $r+1$ to obtain a T_r-parametrization.

We fix an integer r prime to q and we partition $\mathbb{F}_q^\times$ into $\ell = \gcd(r, q-1)$ sets $A_1, \dots, A_\ell$ such that $x \mapsto x^r$ is a bijection from each A_i to $(\mathbb{F}_q^\times)^r$, the set of r-th powers in $\mathbb{F}_q^\times$. We choose representatives $\bar{d}_1, \dots, \bar{d}_\ell$ for cosets of $(\mathbb{F}_q^\times)^r$ and we fix lifts of them, denoted by $d_1, \dots, d_\ell \in \mathcal{O}_L$. For $x \in \mathcal{O}_L \backslash \{0\}$, we set $\xi(x) = d_i$ for i such that $\overline{\mathrm{ac}}(x) \in A_i$.

Now define for $j = (j_1, \dots, j_m) \in \{0, \dots, r-1\}^m$ the function

$$p_{r,j} : (\mathcal{O}_L \backslash \{0\})^m \to (\mathcal{O}_L \backslash \{0\})^m, \quad x = (x_1, \dots, x_m) \mapsto (t^{j_1}\xi(x_1)x_1^r, \dots, t^{j_m}\xi(x_m)x_m^r),$$

where t is our constant symbol for a uniformizer of $\mathcal{O}_L$.

Let $D_{y,i,j} = p_{r,j}^{-1}(P_{y,i}(L))$. By compactness and up to making M larger if necessary, we have that $P_{y,i}(L)$ is a cell around zero. By Hensel's lemma, the union over $j \in \{0, \dots, r-1\}^m$ of the sets $p_{r,j}(D_{y,i,j})$ is equal to $P_{y,i}(L)$. We claim that the family $(\bar{g}_{y,i,j} = g_{y,i} \circ p_{r,j})_{(y,i,j) \in Y(L) \times I \times \{0,\dots,r-1\}^m}$ is the desired T_r-parametrization of $X(L)$. Note that since we used in its definition the lifts d_i, it is an R_r-definable family, where R_r is a set of lifts of representatives of cosets of $(\mathbb{F}_q^\times)^r$. To lighten notations, let us skip for the rest of the proof the subscript (y, i, j). By Lemma 3.4.4 and up to making M larger if necessary, $\bar{g}$ satisfies T_r-approximation on each box contained in D. We show using T_1-approximation for g and ultrametric computations that $\bar{g}$ satisfies T_r-approximation on the whole D.

Fix $x, y \in D$. If x and y are in the same box contained in D, then we are done. Assume then that they are not.

Choose $v \in D$ such that $\overline{\mathrm{ac}}(v_i) = \overline{\mathrm{ac}}(y_i)$ and $|v_i| = |x_i|$, and in the case we moreover have $\overline{\mathrm{ac}}(x_i) = \overline{\mathrm{ac}}(y_i)$, set $v_i = x_i$. Such a v exists by Hensel's lemma and the fact that D is a cell around zero. Define $w \in D$ such that $w_i = v_i$ if $|v_i| = |y_i|$ and $w_i = y_i$ if $|v_i| \neq |y_i|$. We have that w and y lie in the same box contained in D. There are also $d, d', d'' \in \mathcal{O}_L^m$ as prescribed by $p_{r,j}$ such that $\bar{g}(x) = g(dx^r)$, $\bar{g}(w) = g(d'w^r)$ and $\bar{g}(y) = g(d''y^r)$.

We then have

$$\begin{aligned}
|\bar{g}(x) - T_{\bar{g},y}^{<r}(x)| &\leq \max\bigl\{|\bar{g}(x) - \bar{g}(w)|, |\bar{g}(w) - T_{\bar{g},y}^{<r}(w)|, |T_{\bar{g},y}^{<r}(w) - T_{\bar{g},y}^{<r}(x)|\bigr\} \\
&= \max\bigl\{|g(dx^r) - g(d'w^r)|, |\bar{g}(w) - T_{\bar{g},y}^{<r}(w)|, |T_{\bar{g},y}^{<r}(w) - T_{\bar{g},y}^{<r}(x)|\bigr\} \\
&\leq \max\bigl\{|dx^r - d'w^r|, |w-y|^r, |T_{\bar{g},y}^{<r}(w) - T_{\bar{g},y}^{<r}(x)|\bigr\} \\
&\leq \max\bigl\{|x-y|^r, |w-y|^r, |T_{\bar{g},y}^{<r}(w) - T_{\bar{g},y}^{<r}(x)|\bigr\} \\
&\leq \max\bigl\{|x-y|^r, |T_{\bar{g},y}^{<r}(w) - T_{\bar{g},y}^{<r}(x)|\bigr\} \\
&\leq |x-y|^r.
\end{aligned}$$

The first inequality is by the ultrametric triangular inequality, the second is by the global T_1-property for g and the T_r-property on boxes for $\bar{g}$. The third one is because for each i, we have $|d_i x_i^r - d_i' w^r| \le |x_i - y_i|^r$. Indeed, there are three cases to consider. In one case, we have $x_i = w_i$ and $d_i = d_i'$, and then $d_i x_i^r - d_i' w^r = 0$. Or we have $|x_i| \neq |y_i|$. In that case, $|w_i| = |y_i|$ and $|d_i| = |d_1'| \le 1$. Then by the ultrametric property we have $|x_i - y_i| = \max\{|x_i|, |y_i|\}$ and

$$|d_i x_i^r - d_i' w^r| = \max\{|d_i x_i^r|, |d_i' w_i^r|\} \le \max\{|x_i|, |w_i|\}^r = \max\{|x_i|, |y_i|\}^r.$$

The last case is when $|x_i| = |y_i|$ and $\overline{\mathrm{ac}}(x_i) \neq \overline{\mathrm{ac}}(y_i)$. In that case,

$$|w_i| = |x_i|, \qquad \overline{\mathrm{ac}}(w_i) = \overline{\mathrm{ac}}(y_i), \qquad |d_i| = |d_i'| \le 1.$$

We then have $|x_i - y_i| = |x_i|$ and by the choice made in the definition of $p_{r,j}$,

$$\overline{\mathrm{ac}}(d_i x^r) \neq \overline{\mathrm{ac}}(d_i' w^r),$$

whence $|d_i x_i^r - d_i' w^r| = |d_i x^r| \le |x_i|^r = |x_i - y_i|^r$.

The fourth inequality holds because by definition of w, either $w_i = y_i$, or $w_i = x_i$, or $|w_i| = |x_i| = |y_i|$ and $\overline{\mathrm{ac}}(x_i) \neq \overline{\mathrm{ac}}(w_i) = \overline{\mathrm{ac}}(y_i)$. In those three cases, we have $|w_i - y_i| \le |x_i - y_i|$.

To conclude the proof, it remains to prove the last inequality

$$|T_{\bar{g},y}^{<r}(w) - T_{\bar{g},y}^{<r}(x)| \le |x - y|^r.$$

Suppose $T_{\bar{g},y}^{<r}(x) = \sum_{k \in \mathbb{N}^m, |k| < r} c_k (x - y)^k$. For $A \subseteq \mathbb{N}^m$, introduce the notation

$$T_{\bar{g},y}^{<r,A}(x) = \sum_{k \in \mathbb{N}^m,\, |k| < r, k \in A} c_k (x - y)^k.$$

Then set

$$A' = \{k = (k_1, \dots, k_m) \in \mathbb{N}^m \mid k_i = 0 \text{ if } |y_i| \le |x_i - y_i|\},$$

and let A be its complement. The condition can be rephrased by writing that $k_i = 0$ if $w_i \neq x_i$. In particular, for $k \in A'$ we have $(x - y)^k = (w - y)^k$, and hence $T_{\bar{g},y}^{<r,A'}(x) = T_{\bar{g},y}^{<r,A'}(w)$.

Thus it remains to show that

$$|T_{\bar{g},y}^{<r,A}(w) - T_{\bar{g},y}^{<r,A}(x)| \le |x - y|^r.$$

We claim that

$$|T_{\bar{g},y}^{<r,A}(x)| \le |x - y|^r \quad \text{and} \quad |T_{\bar{g},y}^{<r,A}(w)| \le |x - y|^r,$$

which implies the preceding inequality.

Since for each i, $|w_i - y_i| \le |x_i - y_i|$, it is enough to prove that for each $k \in A$ such that $0 < |k| < r$,

$$|c_k (x - y)^k| \le |x - y|^r.$$

From the definition of A, there is some i_0 such that $k_{i_0} > 0$ and $|y_{i_0}| \le |x_{i_0} - y_{i_0}|$. Suppose to lighten the notation that $i_0 = 1$. Set $\underline{r} = (\underline{r}_1, \dots, \underline{r}_m)$ with $\underline{r}_i = k_i$ for $i > 1$ and $\underline{r}_i = r - |k| + k_1 \ge 1$.

Recall the bound for $|c_k|$ obtained from Lemma 3.4.4. We now compute, using this bound and the definition of $\underline{r}$,

$$\begin{aligned}|c_k(x-y)^k| &\le |y|^r_{\min,k}\,|y^k|^{-1}\,|(x-y)^k|\\ &\le |y^{\underline{r}}|\,|y^k|^{-1}\,|(x-y)|^{|k|}\\ &\le |y^{\underline{r}-k}|\,|(x-y)|^{|k|}\\ &= |y_1|^{r-|k|}\,|(x-y)|^{|k|}\\ &\le |x_1-y_1|^{r-|k|}\,|(x-y)|^{|k|}\\ &\le |(x-y)|^{|r|}.\end{aligned}$$

This finishes the proof of the theorem. □

Proof of Theorem 3.1.5. The proof is similar to that of Theorem 3.1.4 above, using Theorem 3.4.2 and then precomposition by power functions. One just needs to delete the application of compactness, and, instead of using the map ξ which chooses and exploits the lifts of cosets of r-th powers in the residue field, one uses parameters from R_r to paste pieces together. (A factor b_r comes in because in this general case the pasting is rougher, since in the residue field, the number of cosets of the r-th powers fails to equal the number of solutions of $x^r = 1$ in general.) The rest of the proof is completely similar. □

4. Points of bounded degree in $\mathbb{F}_q[t]$

4.1. ***A counting theorem.*** The goal of this section is to prove the following theorem, of which Theorem A is a particular case. Recall from the introduction that, for q a prime power and n a positive integer, $\mathbb{F}_q[t]_n$ is the set of polynomials with coefficients in $\mathbb{F}_q$ and degree (strictly) less than n, and, for an affine variety X defined over a subring of $\mathbb{F}_q((t))$, $X(\mathbb{F}_q[t])_n$ denotes the subset of $X(\mathbb{F}_q((t)))$ consisting of points whose coordinates lie in $\mathbb{F}_q[t]_n$. Also, for a subset A of $\mathbb{F}_q((t))^m$, write A_n for the subset of A consisting of points whose coordinates lie in $\mathbb{F}_q[t]_n$.

For an affine (reduced) variety $X \subset \mathbb{A}^m_R$ with R an integral domain contained in an algebraically closed field K, we define the degree of X as the degree of the closure of X_K in $\mathbb{P}^m_K$. For example, if X is a hypersurface given by one (reduced) equation f, then the degree of X equals the (total) degree of f.

Theorem 4.1.1. *Let d, m and δ be positive integers. Then there exist real numbers $C = C(d, m, \delta)$ and $N = N(d, m, \delta)$ such that for each prime $p > N$, each power $q = p^\alpha$ with $\alpha > 0$ an integer, each integer $n > 0$ and each irreducible variety $X \subseteq \mathbb{A}^m_{\mathbb{F}_q((t))}$ of degree δ and dimension d, one has*

$$\#X(\mathbb{F}_q[t])_n \le Cn^2 q^{n(d-1)+\lceil n/\delta\rceil}.$$

We first give a bound for a so-called naive degree. Define the naive degree of a variety $X \subset \mathbb{A}^m_R$ with R an integral domain as the minimum, taken over all tuples of (nonzero) polynomials $f = (f_1, \dots, f_s)$ over R with $X(K) = \{x \in K^m \mid f(x) = 0\}$, of the product of the degrees of the f_i.

Lemma 4.1.2. *Let d, m, and δ be positive integers. Then there exist numbers $C = C(d, m, \delta)$ and $N = N(d, m, \delta)$ such that for each prime $p > N$, each power $q = p^\alpha$ with $\alpha > 0$ an integer, and each* *geometrically irreducible variety $X \subseteq \mathbb{A}^m_{\mathbb{F}_q((t))}$ of degree δ and dimension d, one has that the naive degree of X is bounded by C.*

Proof. From the theory of Chow forms (see [Samuel 1955] or [Catanese 1992]), a variety $X \subseteq \mathbb{A}^m_{\mathbb{F}_q((t))}$ of degree δ and dimension d is determined set-theoretically by a hypersurface of degree δ in the Grasmanniann of $G(m-d-1, m)$ of $(m-d-1)$-dimensional vector subspaces of the m-dimensional space. As explained for example in [Catanese 1992], one can construct from such a hypersurface a system of $m(d+1)$ equations of degrees at most δ such that their zero sets coincide set-theoretically with X. Hence the naive degree of X is bounded by $\delta m(d+1)$. □

The following trivial bound for points of bounded height is typical.

Lemma 4.1.3. *Let d, m and δ be positive integers. Then there exist real numbers $C = C(d, m, \delta)$ and $N = N(d, m, \delta)$ such that for each prime $p > N$, each power $q = p^\alpha$ with $\alpha > 0$ an integer, each integer $n > 0$ and each irreducible variety $X \subseteq \mathbb{A}^m_{\mathbb{F}_q((t))}$ of degree δ and dimension d, one has*

$$\#X(\mathbb{F}_q[t])_n \leq Cq^{nd}.$$

Proof. The lemma follows easily from Noether's normalization lemma and Lemma 4.1.2. □

Let us first reduce the statement of Theorem 4.1.1 to the case of planar curves, similarly to [Pila 1995]. In this section, definable means definable in the language $\mathcal{L}_{\mathrm{DP}}$ of Setting 3.1.1 and with $\mathcal{O} = \mathbb{Z}$.

Reduction of Theorem 4.1.1 to the case $m = 2$ and $d = 1$.. Fix positive integers d, m, δ. By Lemma 4.1.2, irreducible varieties in $\mathbb{A}^m$ of dimension d and of degree δ form a definable family of sets, say, with parameter z in a definable (and Zariski-constructible) set Z; write X_z for the variety in $\mathbb{A}^m$ corresponding to the parameter $z \in Z$. Assume first that $m > 2$ and $d = 1$. Consider the family of linear projections $p_{a,b} : \mathbb{A}^m \to \mathbb{A}^2$ written in coordinates $x = \sum a_i x_i$ and $y = \sum b_i y_i$ and with parameters $(a, b) \in \mathbb{A}^{2m}$. Then, for each $z \in Z$, there is a nonempty Zariski open subset of parameters $O_z \subseteq \mathbb{A}^{2m}$ such that $p_{a,b}$ is surjective and the varieties X_z and $p_{a,b}(X_z)$ have the same degree δ (and are both irreducible of dimension 1) for all $(a, b) \in O_Z$. Clearly the open sets O_z form a definable family of sets with parameter $z \in Z$.

Now suppose that the prime p is large enough and that $q = p^\alpha$ for some α. Since the complement of O_z is of dimension less than $2m$ by Lemma 4.1.3, and since the O_z form a definable family, we can find for each $z \in Z(\mathbb{F}_q((t)))$ a point (a^0, b^0) in $O_z(\mathbb{F}_q[t])_1$ (hence, so to say, a tuple of polynomials in t over $\mathbb{F}_q$ and of degree 0). Hence, p_{a^0,b^0} maps points in $\mathbb{F}_q[t]^m_n$ to points in $\mathbb{F}_q[t]^2_n$. Furthermore, the fibers of p_{a^0,b^0} on X_z are finite, uniformly in z, say, bounded by C. We thus have that for each large enough p, each z in $Z(\mathbb{F}_q((t)))$, and each $n > 0$, that

$$\#X_z(\mathbb{F}_q[t])_n \leq C\#p(X_z)(\mathbb{F}_q[t])_n.$$

Hence the result for $d = 1$ and general $m > 1$ follows from the case $d = 1$ and $m = 2$.

Assume now that $m \geq 2$ and $d > 1$. By a projection argument as above, we can assume that $d = m - 1$. Consider the family of hyperplanes $H = H_{\alpha,b}$ with equation $\sum \alpha_i x_i = b$ and parameters α and b. Then for each $z \in Z$ there is a nonempty Zariski open subset O_z of $\mathbb{A}^{m+1}$ such that if (α, b) lies in O_z, then $X_z \cap H_{\alpha,b}$ is irreducible, of degree δ and dimension d. Hence, similarly as above, for large enough primes p and with $q = p^{\alpha}$, we can find for each z in $Z(\mathbb{F}_q((t)))$ a point (α^0, b^0) in $O_z(\mathbb{F}_q[t])_1$. Now consider the family of hyperplanes H_b of equations $\sum \alpha_i^0 x_i = b$ with parameter b running over $\mathbb{F}_q((t))$. Since (α^0, b^0) belongs to $O_z(\mathbb{F}_q[t])_1$, and by construction, there are at most finitely many values for b such that $(\alpha^0, b) \notin O_z(\mathbb{F}_q((t)))$, say, $b_1, \ldots, b_k$. In any case we can assume that $X_z \cap H_{b_j}$ is of dimension at most $m-1$ for each j, and hence that

$$\#(X_z \cap H_{b_j}) \leq C q^{m-1}$$

for some C which is independent of q and n, by Lemma 4.1.3. To treat the remaining part, we apply the induction hypothesis to $X'_z = (X_z \cap H_b)$ for b outside $\{b_1, \ldots, b_k\}$, and we take the sum of the bounds over all values of b in $\mathbb{F}_q[t]_n$. □

4.2. ***Determinant lemma.*** We fix the following notation for the rest of the paper. For $\alpha = (\alpha_1, \ldots, \alpha_m)$ in $\mathbb{N}^m$, set $|\alpha| = \alpha_1 + \cdots + \alpha_m$. Set also

$$\Lambda_m(k) := \{\alpha \in \mathbb{N}^m \mid |\alpha| = k\}, \qquad \Delta_m(k) := \{\alpha \in \mathbb{N}^m \mid |\alpha| \leq k\},$$
$$L_m(k) := \#\Lambda_m(k), \qquad D_m(k) = \#\Delta_m(k).$$

Lemma 4.2.1 [Cluckers et al. 2015, Lemma 3.3.1]. *Let K be a discretely valued henselian field. Fix $\mu, r \in \mathbb{N}$, and U an open subset of K^m contained in a box that is a product of m closed balls of valuative radius ρ. Fix $x_1, \ldots, x_\mu \in U$, and functions $\psi_1, \ldots, \psi_\mu : U \to K$. Assume that*

- *the integer r satisfies*
$$D_m(r-1) \leq \mu < D_m(r);$$
- *the functions $\psi_1, \ldots, \psi_\mu$ satisfy T_r on U.*

Then

$$\operatorname{ord}_t(\det(\psi_i(x_j))) \geq \rho e,$$

where $e = \sum_{i=0}^{r-1} i L_m(i) + r(\mu - D_m(r-1))$.

4.3. ***Hilbert functions.*** Fix a field K. For s a positive integer, denote $K[x_0, \ldots, x_n]_s$ the space of homogenous polynomials of degree s. Let I be a homogenous ideal of $K[x_0, \ldots, x_n]$, associated to an irreducible variety of dimension d and degree δ of $\mathbb{P}^n_K$. Let $I_s = I \cap K[x_0, \ldots, x_n]_s$ and let $\mathrm{HF}_I(s) = \dim_K K[x_0, \ldots, x_n]_s / I_s$ be the (projective) Hilbert function of I. The Hilbert polynomial HP_I of I is a polynomial such that for s large enough, $\mathrm{HP}_I(s) = \mathrm{HF}_I(s)$. It is a polynomial of degree d and leading coefficient $\delta/d!$.

Fix some monomial ordering in the sense of [Cox et al. 2015]. Denote by $\mathrm{LT}(I)$ the ideal generated by leading terms of elements of I. By [Cox et al. 2015], the Hilbert functions of I and $\mathrm{LT}(I)$ are equal. It follows that

$$\mathrm{HF}_I(s) = \#\{\alpha \in \Lambda_{n+1}(s) \mid x^\alpha \notin \mathrm{LT}(I)\}.$$

Define also for $i = 0, \dots, n$,

$$\sigma_{I,i}(s) = \sum_{\alpha \in \Lambda_{n+1}(s),\, x^\alpha \notin \mathrm{LT}(I)} \alpha_i. \tag{4.3.1}$$

Hence, we have $s\,\mathrm{HF}_I(s) = \sum_{i=0}^{n} \sigma_{I,i}(s)$. The function $\sigma_{I,i}$ is also equal to a polynomial function of degree at most $d+1$, for s large enough. It follows that there exist nonnegative real numbers $a_{I,i}$ such that

$$\frac{\sigma_{I,i}(s)}{s\,\mathrm{HP}_I(s)} = a_{I,i} + O(1/s) \tag{4.3.2}$$

when s goes to $+\infty$.

Remark 4.3.1. The s chosen large enough so that $\mathrm{HF}_I(s)$ is a polynomial and the implicit constant in (4.3.2) depend on I. However, since $\mathrm{HF}_I(s) = \mathrm{HF}_{\mathrm{LT}(I)}$, they in fact only depend on $\mathrm{LT}(I)$. Since they are obtained in a pure combinatorial way, they do not depend on the field K. If we let I vary among ideals generated by a polynomial of degree at most d, then only finitely many different $\mathrm{LT}(I)$ appear. So the previous constants can be chosen uniformly over the whole family of such ideals I.

We will also use the following lemma of Salberger [2007], which is the reason why we will use a projective embedding in the proof of Theorem 4.1.1.

Lemma 4.3.2 [Salberger 2007]. *Let X be a closed equidimensional subscheme of dimension d of $\mathbb{P}^m_K$. Assume that no irreducible component of X is contained in the hyperplane at infinity defined by $x_0 = 0$. Let $<$ be the monomial ordering defined by $\alpha \le \beta$ if $|\alpha| < |\beta|$ or $|\alpha| = |\beta|$ and for some i, $\alpha_i > \beta_i$ and $\alpha_j = \beta_j$ for $j < i$. Then*

$$a_{I,1} + \dots + a_{I,m} \le \frac{d}{d+1}.$$

4.4. *Proof of Theorem 4.1.1 for m = 2 and d = 1.* Fix a positive integer δ. Clearly all irreducible curves in $\mathbb{A}^2$ of degree δ form a definable family, say, with parameter z in a definable (and Zariski-constructible) set Z; write X_z for the curve in $\mathbb{A}^2$ corresponding to the parameter $z \in Z$.

Apply Theorem 3.1.4 to the definable family of the definable sets X_z. It gives some constant C and, for some M, all local fields K in $\mathcal{B}_{\mathbb{Z},M}$ and all integers $r > 0$ prime to q_K, a T_r-parametrization of $X_z(\mathcal{O}_K)$ with Cr many pieces. Fix such a K and a parameter $z \in Z(K)$ corresponding to an irreducible curve $X_z \subset \mathbb{A}^2_K$ of degree δ.

Consider the map

$$\iota : \mathbb{A}^2_K \to \mathbb{A}^3_K, \quad (x, y) \mapsto (1, x, y)$$

and the corresponding embedding

$$\underline{\iota} : \mathbb{A}^2_K \hookrightarrow \mathbb{P}^2_K, \quad (x, y) \mapsto [1 : x : y].$$

Denote by I_z the homogenous ideal associated to the closure of $\underline{\iota}(X_z)$.

Fix some positive integer s, set

$$M_z(s) = \{\alpha \in \Lambda_3(s) \mid x^\alpha \notin \mathrm{LT}(I_z)\}, \quad \mu = \#M_z(s) \quad \text{and} \quad e = \tfrac{1}{2}\mu(\mu-1).$$

Now consider the given T_r-parametrization of $X_z(\mathcal{O}_K)$ with $r = \mu$ and work on one of the $C\mu$ pieces $U_z \subseteq \mathcal{O}_K$ with function $g_z : U_z \to X(\mathcal{O}_K)$ satisfying T_μ on U_z.

Fix a closed ball $B_\beta \subseteq \mathcal{O}_K$ of valuative radius β. Fix some points $y_1, \dots, y_\mu$ in $(g(B_\beta \cap U))_n$ and consider the determinant

$$\Delta = \det(\iota(y_i)^\alpha)_{1 \le i \le \mu, \alpha \in M_z(s)}.$$

Since the composition of functions satisfying T_μ also satisfies T_μ, we can apply Lemma 4.2.1 with $m = 1$, $r = \mu$ to get that

$$\operatorname{ord}_t \Delta \ge \beta e.$$

On the other hand, since the points y_i are of degree less than n as polynomials in t over $\mathbb{F}_{q_K}$, we also have

$$\deg \Delta \le (n-1)(\sigma_1 + \sigma_2),$$

where σ_1, σ_2 are defined by equation (4.3.1). Hence, if Δ is not zero, then

$$\operatorname{ord}_t \Delta \le (n-1)(\sigma_1 + \sigma_2).$$

It follows that $\Delta = 0$ whenever

$$\beta e > (n-1)(\sigma_1 + \sigma_2). \tag{4.4.1}$$

When such an inequality holds, the matrix $A = (y_i^\alpha)$ is of rank less than μ. Fix a minor of maximal rank $B = (y_i^\alpha)_{i \in I, \alpha \in J}$ and some $\alpha_0 \in M_z(s) \setminus J$. Then the polynomial

$$f(x, y) = \det\begin{pmatrix} y_i^\alpha \\ (1, x, y)^\alpha \end{pmatrix}_{i \in I,\, \alpha \in J \cup \{\alpha_0\}}$$

is of total degree at most s and nonzero, since the coefficient of $(1, x, y)^{\alpha_0}$ is $\det(B)$. Moreover, it vanishes at all points in $g(B_\beta \cap U)_n$ but does not vanish on the whole X_z, since its exponents lie in $M_z(s)$ and X_z is irreducible. Hence, by Bézout's theorem, there are at most $s\delta$ points in $(g(B_\beta \cap U))_n$.

We now show how to choose s and β in terms of n such that inequality (4.4.1) holds. Recall that $\mu = \#M_z(s) = \mathrm{HF}_{I_z}(s)$. By properties of Hilbert polynomials and equation (4.3.2), we have

$$\mu = \delta s + O(1) \tag{4.4.2}$$

and

$$\frac{\sigma_i}{\mu} = a_i s + O(1).$$

Here and below, the notation $O(1)$ refers to $s \to +\infty$, and by Remark 4.3.1, the implicit constant is independent of z and q_K. Combining those two equations, we get

$$\sigma_i = a_i \delta s^2 + O(s) \quad \text{and} \quad e = \tfrac{1}{2}\delta^2 s^2 + O(s),$$

and finally, by applying Lemma 4.3.2,

$$\frac{\sigma_1 + \sigma_2}{e} \le \frac{1}{\delta} + O(s^{-1}).$$

Hence there is some s_0 and $C_0 > 0$ such that for every $s \geq s_0$,

$$\frac{\sigma_1 + \sigma_2}{e} \leq \frac{1}{\delta} + C_0 s^{-1}.$$

Recall that the coefficients of Hilbert polynomials can be bounded in terms of the degree of the curve and that the characteristic is assumed to be large. Hence s_0 and C_0 depend only on the degree δ of the curve X_z.

If follows that for

$$s = \lceil \max\{s_0, 2C_0(n-1)\} \rceil, \tag{4.4.3}$$

we have

$$(n-1)\frac{\sigma_1 + \sigma_2}{e} \leq \left\lceil \frac{n}{\delta} \right\rceil.$$

We can thus set $\beta = \lceil n/\delta \rceil$ to satisfy inequality (4.4.1). It follows from the preceding discussion that there are at most $s\delta$ points in $g(B_\beta \cap U)_n$. From equation (4.4.2), we have $\mu \leq \delta s + C_1$, for some constant C_1, and from (4.4.3), that $s \leq C_2 n$ for some constant C_2, with C_i independent of n. Since we need q^β closed balls of valuative radius β to cover $\mathbb{F}_q[\![t]\!] = \mathcal{O}_K$, and since we have a T_μ-parametrization of $X(\mathbb{F}_q[\![t]\!])$ involving $C\mu$ pieces, we find that (after enlarging C) there are at most

$$Cn^2 q^{\lceil n/\delta \rceil}$$

points in $X(\mathbb{F}_q[t])_n$. $\square$

Remark 4.4.1. In the preprint [Bhargava et al. 2017], Sedunova's result [2017] is used to bound the 2-torsion of class groups of function fields over finite fields; see their Theorem 7.1. One can use instead our Theorem 4.1.1 in the special case of Theorem A to obtain a uniform version of their result. We thank Paul Nelson for directing us to the reference [Bhargava et al. 2017].

5. Uniform non-Archimedean Pila–Wilkie counting theorem

In this section we provide uniform versions in the p-adic fields for large p and also in the fields $\mathbb{F}_q((t))$ of large characteristic of several of the main counting results of [Cluckers et al. 2015] (on rational points on p-adic subanalytic sets). To achieve this we use the uniform parametrization result of Theorem 3.1.4. Furthermore, Proposition 5.1.4 is new in all senses, and is a (uniform) non-Archimedean variant of recent results of [Cluckers et al. 2020; Binyamini and Novikov 2019]; it should be put in contrast with Proposition 4.1.3 of [Cluckers et al. 2015].

5.1. ***Hypersurface coverings.*** We begin by fixing some terminology.

Consider the language $\mathcal{L} = \mathcal{L}_{\mathrm{DP}}^{\mathrm{an}}$ as described in Setting 3.1.1. From now on we only consider definable sets which are subsets of the Cartesian powers of the valued field sort (sometimes in a concrete $\mathcal{L}$-structure, and sometimes for the theory $\mathcal{T}$).

Definition 5.1.1. Let K be an $\mathcal{L}$-structure. An $\mathcal{L}(K)$-definable set $X \subset K^n$ is said to be of dimension d at $x \in X$ if for every small enough box containing x, $X \cap B$ is of dimension d. An $\mathcal{L}(K)$-definable set $X \subset K^n$ is said to be of pure dimension d if it is of dimension d at all points x in $X(K)$.

For an $\mathcal{L}(K)$-definable set $X \subset K^n$, define the algebraic part X^{alg} of X to be the union of all quantifier-free $\mathcal{L}_{\mathrm{DP}}(K)$-definable sets of pure positive dimension and contained in X. Note that the set X^{alg} is in general neither semialgebraic nor subanalytic.

By subanalytic we mean from now on $\mathcal{L}$-definable, or $\mathcal{L}(K)$-definable if we are in a fixed $\mathcal{L}$-structure, and we speak about definable families in the sense explained just below Notation 3.1.3. Likewise, by semialgebraic we mean definable in the language $\mathcal{L}_{\mathrm{DP}}$, or $\mathcal{L}_{\mathrm{DP}}(K)$-definable if we are in a fixed structure (see Setting 3.1.1). Write $\mathcal{T}$ for $\mathcal{T}_{\mathrm{DP}}^{\mathrm{an}}$.

Remark 5.1.2. Observe that the definition of the algebraic part is insensitive to having or not having algebraic Skolem functions on the residue field. Indeed, its definition is local and allows parameters from the structure.

If $x \in \mathbb{Z}$, set $H(x) = |x|$, the absolute value of x. If $x = (x_1, \dots, x_n) \in \mathbb{Z}^n$, set $H(x) = \max_i\{H(x_i)\}$. If L is a local field of characteristic zero, $B \geq 1$ and $X \subseteq L^n$, we set

$$X(\mathbb{Z}, B) = \{x \in X \cap \mathbb{Z} \mid H(x) \leq B\}.$$

If $x \in \mathbb{F}_q[t]$, we set

$$H(x) = q^{\deg_t(x)},$$

where $\deg_t(x)$ is the degree in t of the polynomial x over $\mathbb{F}_q$. For $x = (x_1, \dots, x_n) \in (\mathbb{F}_q[t])^n$, put $H(x) = \max_i\{H(x_i)\}$. We now set for $X \subseteq \mathbb{F}_q[\![t]\!]$ and $B \geq 1$

$$X(\mathbb{F}_q[t], B) = \{x \in X \cap \mathbb{F}_q[t] \mid H(x) \leq B\}.$$

Recall the notation at the beginning of Section 4.1. For all integers d, n, m, set $\mu = D_n(d)$ and let r be the smallest integer such that $D_m(r-1) \leq \mu < D_m(r)$. Then set $V = \sum_{k=0}^{d} kL_n(k)$ and $e = \sum_{k=1}^{r-1} kL_m(k) + r(\mu - D_m(r-1))$.

The following result refines Lemma 4.1.2 of [Cluckers et al. 2015] and has a similar proof.

Lemma 5.1.3. *For all integers d, n, m with $m < n$, consider the integers r, V, e as defined above. Fix a local field L, a subset $U \subseteq \mathcal{O}_L^m$, an integer H and maps $\psi = (\psi_1, \dots, \psi_n) : U \to \mathcal{O}_L^n$ that satisfy T_r-approximation. Then if L is of characteristic zero, the set $\psi(U)(\mathbb{Z}, H)$ is contained in at most*

$$q^m(\mu!)^{m/e}H^{mV/e}$$

hypersurfaces of degree at most d. If L is of positive characteristic, the set $\psi(U)(\mathbb{F}_q[t], H)$ is contained in at most

$$q^m H^{mV/e}$$

hypersurfaces of degree at most d. Moreover, when d goes to infinity, mV/e goes to 0.

Proof. We use the notation introduced at the beginning of Section 4.1. Under the hypothesis of the lemma, fix a closed box $B \subseteq \mathcal{O}_L^m$ of valuative radius α. Then fix points $P_1, \dots, P_\mu \in \psi(B \cap U)(\mathbb{Z}, H)$ (or $\psi(B \cap U)(\mathbb{F}_q[t], H)$) and consider $x_i \in B \cap U$ such that $\psi(x_i) = P_i$. Consider the determinant $\Delta = \det((\psi(x_i)^j)_{1 \leq i \leq \mu,\, j \in \Delta_n(d)}$. Since ψ satisfies T_r-approximation, Lemma 4.2.1 gives $\operatorname{ord}(\Delta) \geq \alpha e$.

In the positive characteristic case, since the P_i are in $\mathbb{F}_q[t]$ of degree less than or equal to $\log_q(H)$, if $\Delta \neq 0$, then $\mathrm{ord}(\Delta) \leq \log_q(H)V$. Hence if $\alpha > \log_q(H)V/e$, then $\Delta = 0$.

In the characteristic zero case, since the P_i are in $\mathbb{Z}$ of height at most H, it follows that $\Delta \in \mathbb{Z}$ is of (Archimedean) absolute value at most $\mu! H^V$. If $\Delta \neq 0$, this implies that $\mathrm{ord}(\Delta) \leq \log_q(\mu! H^V)$. Hence if $\alpha > \log_q(\mu! H^V)/e$, then $\Delta = 0$.

We now assume that α is chosen such that $\Delta = 0$. As in the Bombieri–Pila case, by considering minors of maximal rank, we can produce a hypersurface D of degree d such that all the P_i are contained in D. See the proof of Theorem 4.1.1 for details.

Since we need $q^{m\alpha}$ boxes of radius α to cover $\mathcal{O}_L^m$, in the characteristic zero case, we find that we can cover $\psi(U)(\mathbb{Z}, H)$ by $q^m \mu!^{m/e} H^{mV/e}$ hypersurfaces of degree d. In the positive characteristic case, we can cover $\psi(U)(\mathbb{F}_q[t], H)$ by $q^m H^{mV/e}$ hypersurfaces of degree at most d.

By an explicit computation (see [Pila 2004, p. 212]), we get $e \sim_d C_1(m,n) d^{n+n/m}$ and $V \sim_d C_2(m,n) d^{n+1}$, the equivalences being for $d \to +\infty$. Since $m < n$, mV/e goes to zero as $d \to +\infty$. □

Proposition 5.1.4. *Let integers $m \geq 0$ and $n > m$ be given. Let $X = (X_y)_{y \in Y} \subseteq (\mathrm{VF}^n)_{y \in Y}$ be an $\mathcal{L}$-definable family of subanalytic sets with X_y of dimension m in each model K of $\mathcal{T}$ and each y in $Y(K)$. Then there are a constant $C(X)$ depending only on X, a constant $C'(n, m)$ depending only on n and m, and an integer $N = N(X)$ such that for each $H \geq 2$ and each local field $L \in \mathscr{C}_{\mathcal{O},N}$, the following holds.*

For $y \in Y(L)$ and $H \geq 2$, the set $X_y(L)(\mathbb{Z}, H)$ (or $X_y(L)(\mathbb{F}_{q_L}[t], H)$ for the positive characteristic case) is covered by at most

$$C(X) q_L^m \log(H)^\alpha$$

hypersurfaces of degree at most $C'(n, m) \log(H)^{m/(n-m)}$.

Moreover, we have $\alpha = nm/((m-1)(n-m))$ if $m > 1$ and $\alpha = n/(n-1)$ if $m = 1$.

Proof. We work inductively on m. The case $m = 0$ is clear, as the cardinality of the fibers is then uniformly bounded in y. Assume now $1 \leq m$. Apply the parametrization Theorem 3.1.4 to the definable family X.

We keep the notation from the proof of Lemma 5.1.3. Choose d as a function of H such that $H^{mV/e}$ is bounded (say by 2). From the computations at the end of the proof of Lemma 5.1.3, we can choose $d \sim_H C'(m, n) \log(H)^{m/(n-m)}$.

We have $\mu \sim_H C_3(n, m) d^n$, and since r is the smallest integer such that $D_m(r-1) \leq \mu < D_m(r)$, we have that if $m > 1$, then $r = O_H(\mu^{1/(m-1)})$ and if $m = 1$, then $r = \mu$. From Theorem 3.1.4, we find a T_r-parametrization of X involving $C(X) r^m$ pieces. From Lemma 5.1.3, the points of height at most H on one of the pieces are included in at most $q_L^m (\mu!)^{m/e} H^{mV/e}$ (if $L \in \mathcal{A}_{\mathcal{O}}$) or $q_L^m H^{mV/e}$ (if $L \in \mathcal{B}_{\mathcal{O}}$) hypersurfaces of degree at most d. From the Stirling formula, we see that $(\mu!)^{m/e}$ is bounded. Hence overall, up to enlarging $C(X)$, we find that $X_y(L)(\mathbb{Z}, H)$ or $X_y(L)(\mathbb{F}_{q_L}[t], H)$ is contained in

$$C(X) q_L^m \log(H)^\alpha$$

hypersurfaces of degree at most $C'(n, m) \log(H)^{m/(n-m)}$, with $\alpha = nm/((m-1)(n-m))$ if $m > 1$ and $\alpha = n/(n-1)$ if $m = 1$. □

5.2. ***Blocks.*** In this final section, we give uniform versions of results of [Cluckers et al. 2015, Section 4.2] for local fields of large residue characteristic, in particular of Theorems 4.2.3 and 4.2.4 of [Cluckers et al. 2015]. We thus obtain analogs of Pila–Wilkie counting results, uniformly for local fields of large enough positive characteristic. We leave proofs, which are analogous to the ones for Theorems 4.2.3 and 4.2.4 of [Cluckers et al. 2015], to the reader.

Definition 5.2.1. A subset $W \subset K^m$, with K an $\mathcal{L}$-structure, is called a block if it is either a singleton or a smooth subanalytic set of pure dimension $d > 0$ contained in a smooth semialgebraic set of pure dimension d.

A family of blocks $W \subseteq \mathrm{VF}^{m+s}$, with parameters running over VF^s, is a subanalytic set W such that there exists an integer $s' \geq 0$ and a semialgebraic set $W' \subseteq \mathrm{VF}^{m+s'}$ such that for each model K of $\mathcal{T}$, for each $y \in K^s$ there is a $y' \in K^{s'}$ such that both $W_y(K)$ and $W'_{y'}(K)$ are smooth of the same pure dimension and such that $W_y(K) \subseteq W'_{y'}(K)$.

Note that if W is a block of positive dimension, then $W = W^{\mathrm{alg}}$.

Note that our notion of family of blocks, which corresponds to the one in [Chambert-Loir and Loeser 2017], is a strengthening of the one in [Cluckers et al. 2015] which solely ask that a family of blocks W is such that W_y is a block for each $y \in Y$. However, all the results in Section 4.2 of [Cluckers et al. 2015] hold with this strengthened definition.

Let L be in $\mathcal{A}_{\mathcal{O}}$ and let $k > 0$ be an integer. We define the k-height of $x \in L$ as

$$H_k(x) = \min_a \Big\{ H(a) \;\Big|\; a = (a_1, \dots, a_k) \in \mathbb{Z}^k, \sum_{i=0}^{k} a_i x^i = 0, a \neq 0 \Big\}$$

and for $x = (x_1, \dots, x_n) \in L^n$, $H_k(x) = \max_i \{H(x_i)\}$.

Let $L \in \mathcal{B}_{\mathcal{O}}$ and $k > 0$ be an integer. We define the k-height of $x \in L$ as

$$H_k(x) = \min_a \Big\{ H(a) \;\Big|\; a = (a_1, \dots, a_k) \in \mathbb{F}_{q_L}[t]^k, \sum_{i=0}^{k} a_i x^i = 0, a \neq 0 \Big\}$$

and for $x = (x_1, \dots, x_n) \in L^n$, $H_k(x) = \max_i \{H(x_i)\}$.

If $X \subseteq L^n$, we set

$$X(k, H) = \{x \in X \mid H_k(x) \leq H\}.$$

The following result is a generalized and uniform version of Theorems 4.2.3 and 4.2.4 of [Cluckers et al. 2015].

Theorem 5.2.2. *Let $X = (X_y)_{y \in Y} \subseteq (K^n)_{y \in Y}$ be a subanalytic family of subanalytic sets of dimension $m < n$ in each model of $\mathcal{T}$. Fix $\varepsilon > 0$. Then there are a positive constant $C(X, k, \varepsilon)$, integers $l = l(X, k, \varepsilon)$, $N = N(X, k, \varepsilon)$, $\alpha = \alpha(m, n, k)$, and a family of blocks $W = (W_{y,s})_{(y,s) \in Y \times K^l} \subseteq K^n \times Y \times K^l$ such that the following holds.*

For each $L \in \mathscr{C}_{\mathcal{O},N}$, $H \geq 1$ *and* $y \in Y(L)$, *there is a subset* $S = S(X, k, L, H, y) \subseteq K^s$ *of cardinality at most* $C(X, \varepsilon)q^\alpha H^\varepsilon$ *such that*

$$X_y(L)(k, H) \subseteq \bigcup_{s \in S} W_{y,s}.$$

In particular, if we denote by W_y^ε *the union over* $s \in S$ *of the* $W_{y,s}(L)$ *of positive dimension, we have* $W_y^\varepsilon \subseteq X_y(L)^{\mathrm{alg}}$ *and*

$$\#(X_y(L) \backslash W_y^\varepsilon)(k, H) \leq C(X, \varepsilon)q^\alpha H^\varepsilon.$$

The proof of Theorem 5.2.2 is completely similar to those of [Cluckers et al. 2015, Section 4.2] (namely to the proofs of Proposition 4.2.2 and Theorems 4.2.3 and 4.2.4), where instead of using [Cluckers et al. 2015, Proposition 4.2], one uses Proposition 5.1.4. We skip the proofs and refer to [Cluckers et al. 2015] for details. Theorem B in the introduction is the particular case of Theorem 5.2.2 when $k = 2$.

Remark 5.2.3. Note also that the bound in Proposition 5.1.4 is polylogarithmic, whereas the bound of [Cluckers et al. 2015, Proposition 4.2] is subpolynomial. However, this improvement does not guarantee a polylogarithmic bound in the counting theorems. As in the o-minimal case, such a bound is not expected to hold in general, but might be true in some specific situations, similar to the context of Wilkie's conjecture for $\mathbb{R}^{\exp}$-definable sets.

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